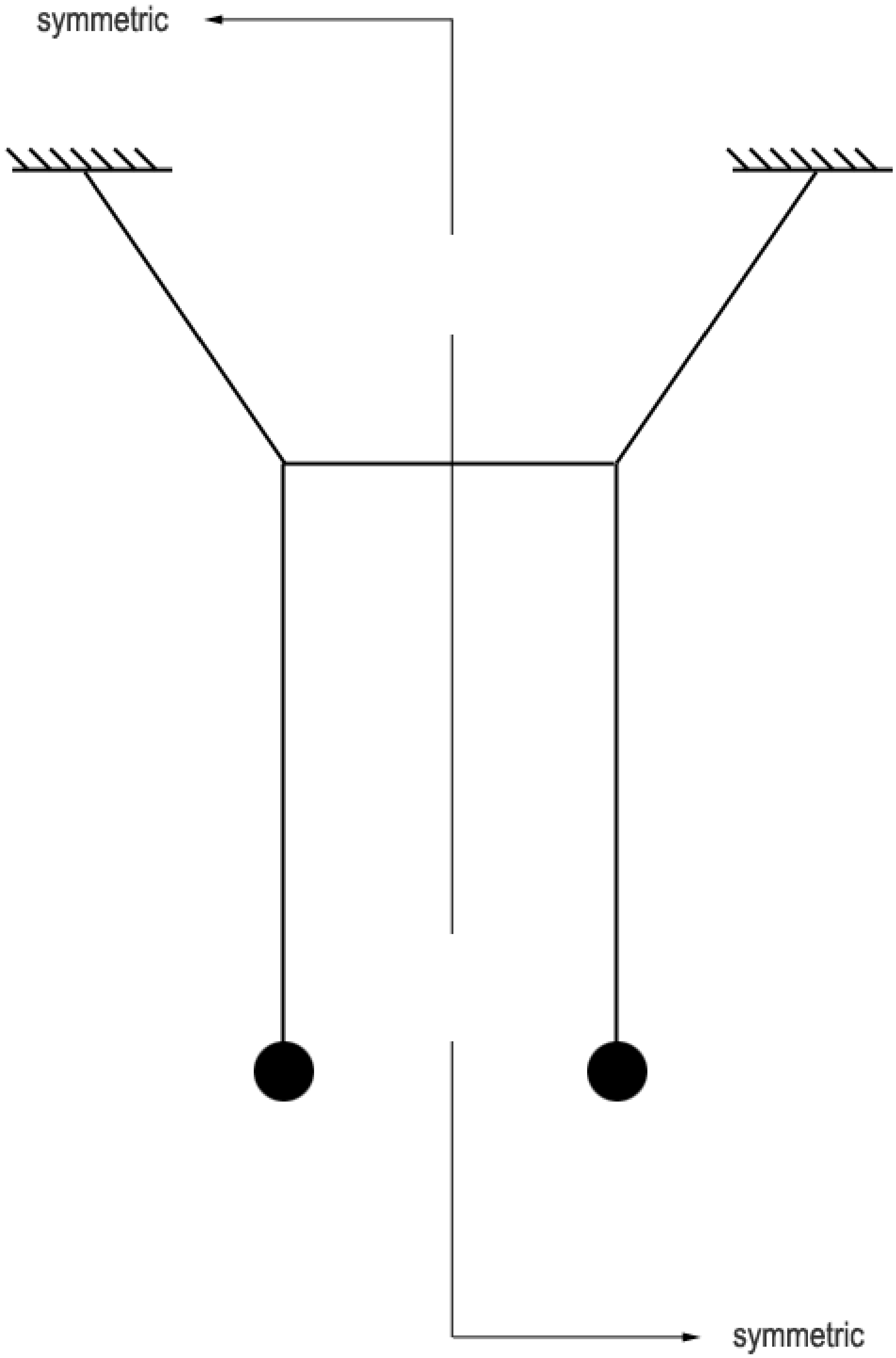
AN ESSAY ON THE STRING-COUPLED PENDULUM



THE STRING-COUPLED PENDULUM

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Author’s preface. The problem of predicting the motion of a string-coupled pendulum, shown opposite, was presented to my freshman college physics class as unsolved. And so it remained, at least in my thinking, for the next ten years. This is the season in which a youth might imbibe any number of Western man’s elegant and related intellectual syntheses. These include our model of light pulses as orthogonally alternating electric and magnetic fields, which measure the universe by the speed of their advance; or the eccentrically periodic ascendance and decline of pagan and puritan instincts that we record as human history; or the endless precession of DNA's twin coils, carrying the history of life's evolution. Each of these phenomena manifests what we would call the working-out of a dualistic principle with time.

Fascination with the string-coupled pendulum would seem to owe to its being the simplest imaginable generator of this sort of movement. Even while watching the pendulum's motion, it is hard to believe that such a simple device could possibly behave in ways that are analogous to the several complex and important processes of which it might be thought symbolic. Perhaps it is a respect for the imprint that this image of dualism-in-motion has made on recent thought that has led to establishing the mechanics of a string-coupled pendulum.

A notarized memorial attached to my original solution indicates that it was submitted in 1975. Though I do not recall the journal, their rejection was unforgettable insofar as I was elaborately informed that I had not defined the problem – specifically in that I did not distinguish the horizontal member as to whether it was a string or a rigid mass-less beam. One might have thought the *string-coupled* aspect title might suffice, since the distinction has no bearing on the motions observed: a beam, however rigid, cannot develop a bending moment when the only forces applied to it are tensile and act at its ends. Absent a bending moment, rigidity is irrelevant.

Many years later, in the course of an email discussion of my solution, I was to learn that another solution by a Professor Michael J. Moloney was published in the American Journal of Physics (Volume 46, Issue 12, pp. 1245-1246; 1978). Though this submission's completeness, dimensional integrity, and initial assumptions are questionable to me, Professor Moloney does not acknowledge such questions, and thereby retains the right of precedence.

This more academically acceptable solution remains problematic in my thinking because it only examines motions that are perpendicular to the plane in which the resting pendulum hangs. As is shown in this paper, string-coupled pendula exhibit their most interesting dynamics when set in motion by initial horizontal displacements in any direction.

Professor Moloney's analytic premise has motion directed by dynamic, horizontal torsion. But we observe that the pendulum's characteristic bi-harmonic oscillations proceed quite clearly from an initial displacement of one pendulum weight completely within the resting plane – whence it is impossible for torsion to develop. Noting that the pendulum's behaviors exhibit a continuous pattern as the initial displacement varies from perpendicular to the resting plane to being confined wholly within it, we reject the premise of torsion as having much analytic potency.

In any case, this paper revises a digital scan of my 1975 original. The revision was undertaken during odd, idle hours in order to exploit the possibilities that desktop publishing software has for creating appealing copy. It is resubmitted for whatever uses and amusements it might provide.

Kurt Roemer
San Francisco
June, 2013

An Essay on the String-coupled Pendulum

Article 1: Notation, co-ordinate frames, and sign conventions. When at rest, a symmetric string-coupled pendulum can be fully described by as few as four parameters, viz.: the Q, L, S and H of Figure 1-1.

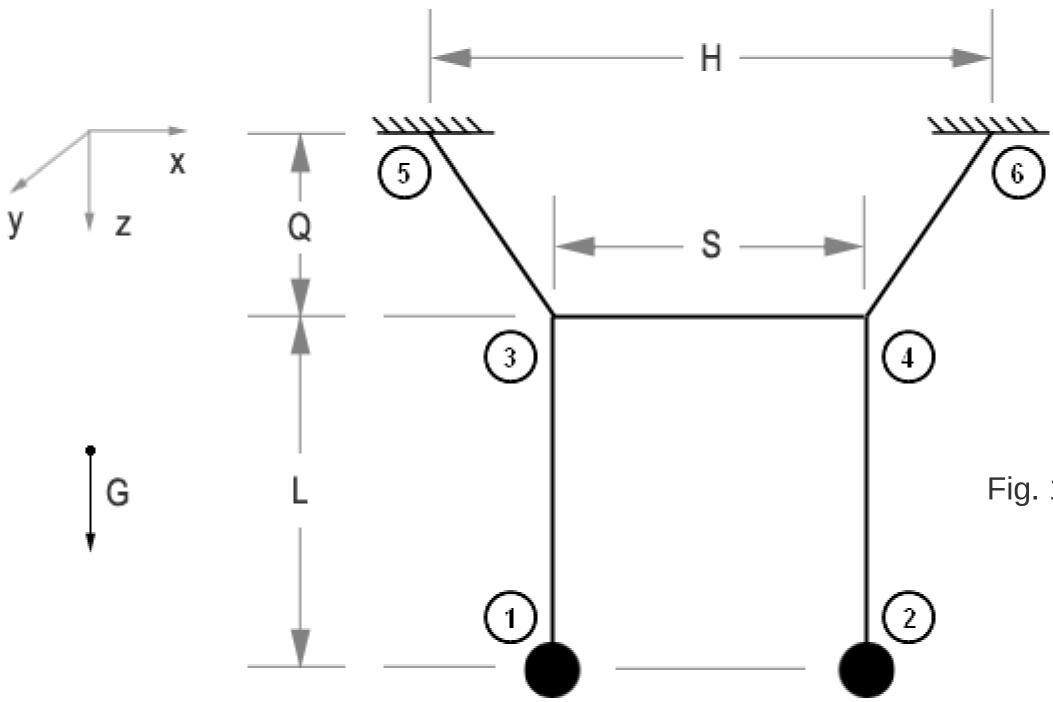


Fig. 1-1

These dimensions lie in the figure's 'X-Z' plane, where the 'X' direction is given by a line joining the two support locations of the pendulum, and the 'Z' direction is aligned with gravity G. The 'Y' direction is perpendicular to this plane, and the positive sense or each direction is chosen so as to give us a right-handed frame of reference.

Circled integers ① through ⑥ refer to this system's node points. In all our analysis we will refer to the variables of position P acceleration A and reaction R as they occur at these nodes and as they act in each of the three coordinate directions, e.g.: P_{1X}. A variable M denotes the mass of either pendulum weight.

Article 2: The equation of motion for a bi-harmonic oscillator. A string-coupled pendulum's most characteristic dynamics are those of bi-harmonic oscillation, as instantiated by Figure 2-1.

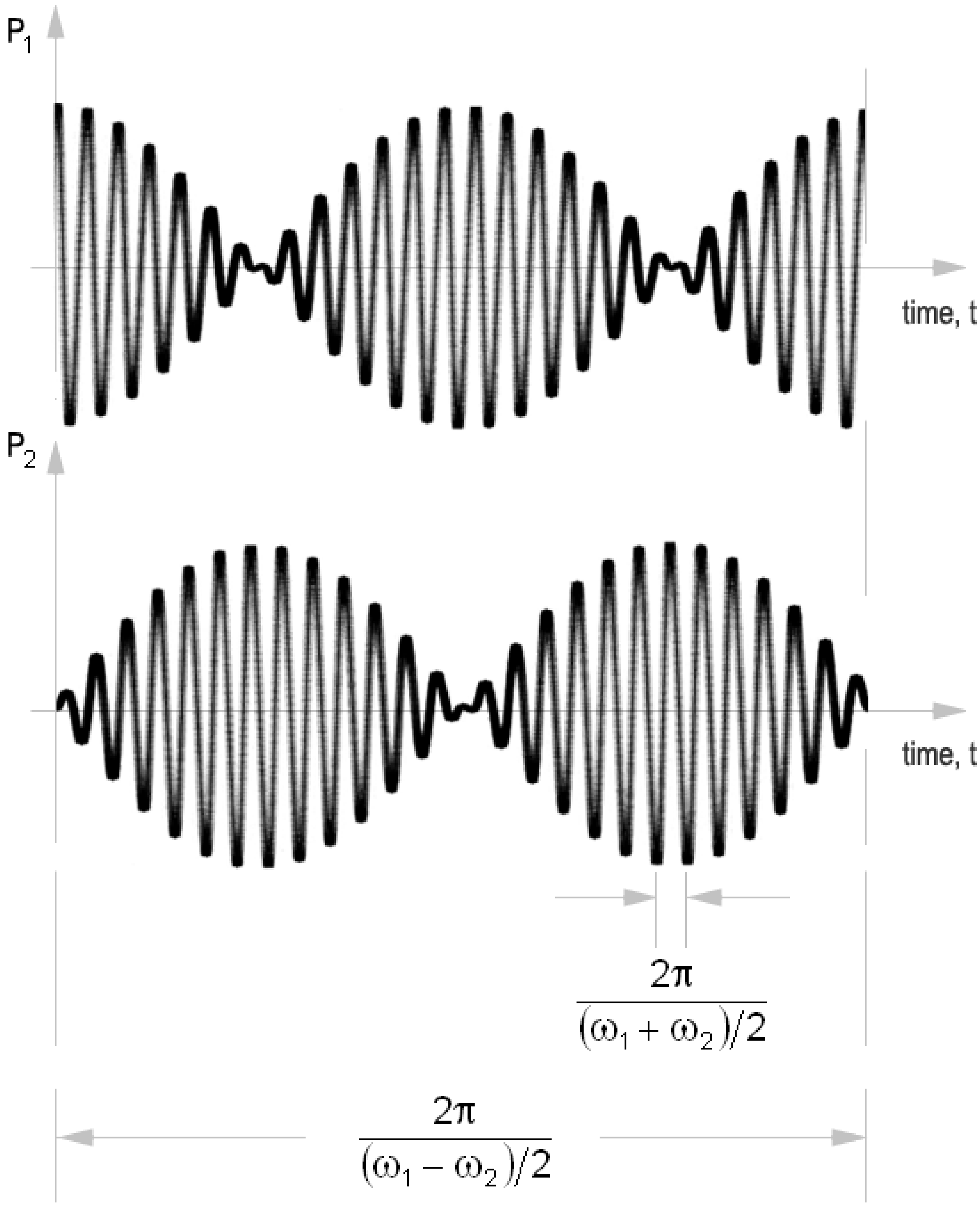


Fig. 2-1

These dynamics will be observed when the pendulum's movement is initiated by displacing one mass along either the X- or the Y-axis. A general algebraic description of the two wave forms shown in Figure 2-1 would be …

$$P_1 = D \cdot \cos\left[\left(\frac{\omega_1 + \omega_2}{2}\right) \cdot t\right] \cdot \cos\left[\left(\frac{\omega_1 - \omega_2}{2}\right) \cdot t\right] \qquad \text{Eqa. 2-1}$$

$$P_2 = D \cdot \sin\left[\left(\frac{\omega_1 + \omega_2}{2}\right) \cdot t\right] \cdot \sin\left[\left(\frac{\omega_1 - \omega_2}{2}\right) \cdot t\right] \qquad \text{Eqa. 2-2}$$

where D corresponds to the value of P_1 when time t is equal to zero; where P_2 is zero at t=0; and where ω_1 and ω_2 define the periods of the bi-harmonic cycles and epicycles marked-out in the lower portion of Figure 2-1. Solution to the pendulum problem lies in establishing these internal frequencies ω_1 and ω_2 on the basis of the dimensions describing a particular string-coupled pendulum. Let us proceed to isolate the pendulum's natural frequencies empirically before presenting a rigorous solution – which requires more algebra than would be desirable for satisfying an ordinary level of curiosity.

If (in the analysis to follow) we are able to establish that the mutual effects of the X- and Y-movements of the pendulum can be superimposed upon one another by simple addition, then Equations 2-1 and 2-2 can be expected to govern motions proceeding from all initiating displacements D_{1X}, D_{2X}, D_{1Y}, and D_{2Y} of the two masses.

Retaining for the moment our reference to motions occurring entirely in the X- or entirely in the Y-direction, the general equations for the movements along either coordinate axis would then be:

$$\begin{aligned} P_1 = {} & D_1 \cdot \cos\left[\left(\frac{\omega_1 + \omega_2}{2}\right) \cdot t\right] \cdot \cos\left[\left(\frac{\omega_1 - \omega_2}{2}\right) \cdot t\right] \\ & + D_2 \cdot \sin\left[\left(\frac{\omega_1 + \omega_2}{2}\right) \cdot t\right] \cdot \sin\left[\left(\frac{\omega_1 - \omega_2}{2}\right) \cdot t\right] \end{aligned} \qquad \text{Eqa. 2-3}$$

$$\begin{aligned} P_2 = {} & D_2 \cdot \cos\left[\left(\frac{\omega_1 + \omega_2}{2}\right) \cdot t\right] \cdot \cos\left[\left(\frac{\omega_1 - \omega_2}{2}\right) \cdot t\right] \\ & + D_1 \cdot \sin\left[\left(\frac{\omega_1 + \omega_2}{2}\right) \cdot t\right] \cdot \sin\left[\left(\frac{\omega_1 - \omega_2}{2}\right) \cdot t\right] \end{aligned} \qquad \text{Eqa. 2-4}$$

Intuition would suggest that setting $D_1 = D_2$ or $D_1 = -D_2$ will produce some interesting results. Let us see.

When $D_1 = D_2$ (=D) obtains, Equations 2-3 and 2-4 become:

$$P_1 = P_2 = D \cdot \left\{ \cos\left[\left(\frac{\omega_1 + \omega_2}{2}\right) \cdot t\right] \cdot \cos\left[\left(\frac{\omega_1 - \omega_2}{2}\right) \cdot t\right] + \sin\left[\left(\frac{\omega_1 + \omega_2}{2}\right) \cdot t\right] \cdot \sin\left[\left(\frac{\omega_1 - \omega_2}{2}\right) \cdot t\right] \right\} \qquad \text{Eqa. 2-5}$$

Now we can apply the formulae for the sums and differences of angles to the expression in the brackets of this equation. Recalling that …

$$\cos(\theta + \phi) = \cos(\theta) \cdot \cos(\phi) - \sin(\theta) \cdot \sin(\phi); \qquad \cos(-\theta) = \cos(\theta)$$

$$\sin(\theta + \phi) = \sin(\theta) \cdot \cos(\phi) + \cos(\theta) \cdot \sin(\phi); \qquad \sin(-\phi) = -\sin(\phi)$$

we can transform Equation 2-5 into:

$$\begin{aligned} P_1 = P_2 = D \bullet \Bigg\{ & \left[\cos\left(\frac{\omega_1}{2} \cdot t\right) \cdot \cos\left(\frac{\omega_2}{2} \cdot t\right) - \sin\left(\frac{\omega_1}{2} \cdot t\right) \cdot \sin\left(\frac{\omega_2}{2} \cdot t\right)\right] \\ & \bullet \left[\cos\left(\frac{\omega_1}{2} \cdot t\right) \cdot \cos\left(\frac{\omega_2}{2} \cdot t\right) + \sin\left(\frac{\omega_1}{2} \cdot t\right) \cdot \sin\left(\frac{\omega_2}{2} \cdot t\right)\right] \\ & + \left[\sin\left(\frac{\omega_1}{2} \cdot t\right) \cdot \cos\left(\frac{\omega_2}{2} \cdot t\right) + \cos\left(\frac{\omega_1}{2} \cdot t\right) \cdot \sin\left(\frac{\omega_2}{2} \cdot t\right)\right] \\ & \bullet \left[\sin\left(\frac{\omega_1}{2} \cdot t\right) \cdot \cos\left(\frac{\omega_2}{2} \cdot t\right) - \cos\left(\frac{\omega_1}{2} \cdot t\right) \cdot \sin\left(\frac{\omega_2}{2} \cdot t\right)\right] \Bigg\} \end{aligned} \qquad \text{Eqa. 2-6}$$

Multiplying as indicated brings us to:

$$\begin{aligned} P_1 = P_2 = D \bullet \Bigg\{ & \cos^2\left(\frac{\omega_1}{2} \cdot t\right) \cdot \cos^2\left(\frac{\omega_2}{2} \cdot t\right) - \sin^2\left(\frac{\omega_1}{2} \cdot t\right) \cdot \sin^2\left(\frac{\omega_2}{2} \cdot t\right) \\ & + \sin^2\left(\frac{\omega_1}{2} \cdot t\right) \cdot \cos^2\left(\frac{\omega_2}{2} \cdot t\right) - \cos^2\left(\frac{\omega_1}{2} \cdot t\right) \cdot \sin^2\left(\frac{\omega_2}{2} \cdot t\right) \Bigg\} \end{aligned} \qquad \text{Eqa. 2-7}$$

Factoring with an eye to $\cos^2(\omega_2 \cdot t/2) + \sin^2(\omega_2 \cdot t/2) = 1$ will yield:

$$P_1 = P_2 = D \cdot \left\{ \cos^2\left(\frac{\omega_2}{2} \cdot t\right) - \sin^2\left(\frac{\omega_2}{2} \cdot t\right) \right\} \qquad \text{Eqa. 2-8}$$

Observing that $\cos^2\omega - \sin^2\omega = \cos(2\omega)$ we finally arrive at …

$$P_1 = P_2 = D \cdot \cos(\omega_2 \cdot t) \qquad \text{Eqa. 2-9}$$

which isolates the smaller of the two internal frequencies ω_2 governing the simple harmonic motion proceeding from equal initial displacements, $D_1 = D_2$, of both masses in either the X- or the Y- direction. An equally tedious process, employing the same trigonometric identities in the same way, will show the equal and opposite initial displacements of the pendulum's two masses, $D_1 = -D_2$, will yield a similar equation in the larger of the internal frequencies, ω_1:

$$P_1 = -P_2 = D \cdot \cos(\omega_1 \cdot t) \qquad \text{Eqa. 2-10}$$

Now let us bring Equations 2-9 and 2-10 to bear on the movements to be observed along the X- and Y- axes following either of these sets of initial conditions.

Article 3: Observations of motion along the Y axis. Taking the case of equal initial displacements along the Y-axis first, our observations would be those depicted in Figure 3-1, where the two masses behave identically to the single mass of a simple pendulum with a total string length of Q + L. The equation for the motion of a simple pendulum, such as on the right in Figure 3-1,

$$P = D \cdot \cos(\omega \cdot t) \qquad \text{Eqa. 3-1}$$

will be derived later as a part of our formal statement of the mechanics of the string-coupled pendulum. It will then be shown that the frequency ω of a simple pendulum is given by the square root of the intensity of the local gravity field G divided by the length E of the string joining the pendulum weight to its support:

$$\omega = \sqrt{G/E} \qquad \text{Eqa. 3-2}$$

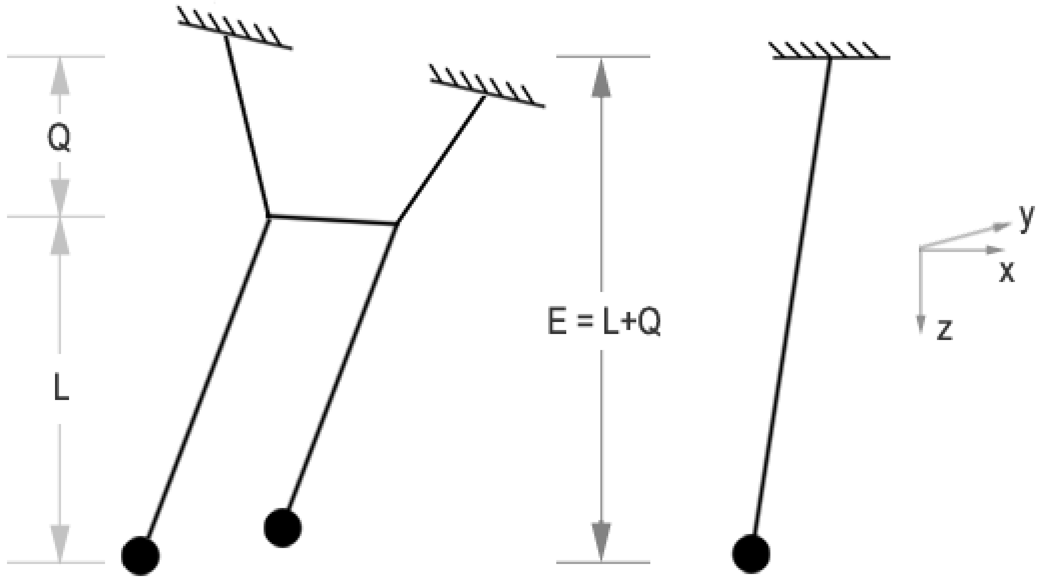


Fig. 3-1

A comparison of Equations 2-9 and 3-1, together with our analogy to the simple pendulum in Figure 3-1, allows us to transform Equation 3-2 into the equation for the smaller of the two frequencies ω_{2Y} governing motion along the Y-axis of a string-coupled pendulum:

$$\omega_{2Y} = \sqrt{\frac{G}{L+Q}} \qquad \text{Eqa. 3-3}$$

This analogy to a simple pendulum of an ‘appropriate equivalent length’ E is also helpful in determining the larger of the two frequencies governing motion along the Y-axis. Recalling from Equation 2-10 that ω_{1Y} is the frequency of the simple harmonic motion following from equal and opposite initializing displacements, we develop Figure 3-2 to visualize the ensuing motion by having the pendulum weights move in a manner that always keeps them 180º out of phase. The period of this motion is that of a simple pendulum with an equivalent length E somewhere between L and L + Q:

$$L \leq E \leq L+Q \qquad \text{Eqa. 3-4}$$

In order to determine E for the initial conditions in effect here, we must examine the extreme cases of the value of S, the length of the horizontal string joining the two halves of a string-coupled pendulum.

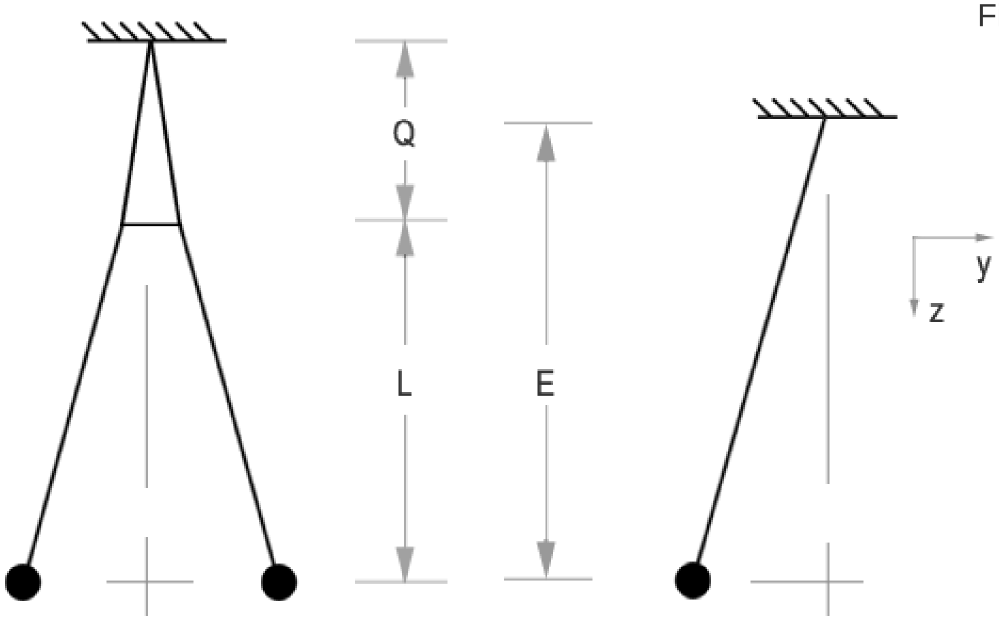


Fig. 3-2

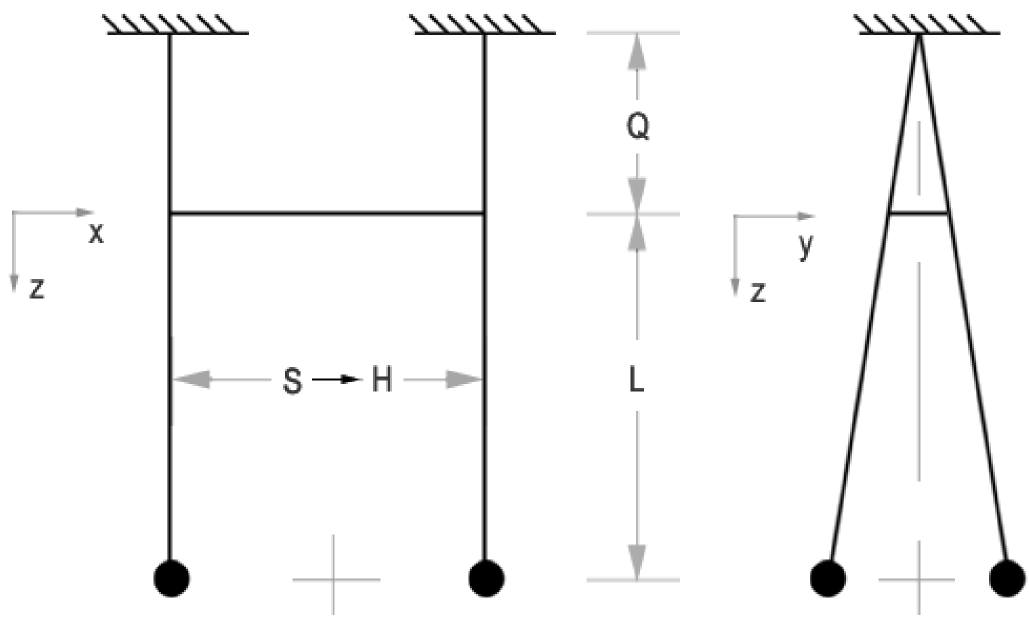


Fig. 3-3

As shown in Figure 3-3, when S approaches H the two halves of the pendulum become increasing independent, and the inner and outer cycles merge to a single harmonic wave form. Here the simple pendulum producing analogous motion has an equivalent length E equal to L + Q:

$$E = L + Q \quad \text{for} \quad S \rightarrow H \qquad \text{Eqa. 3-5}$$

Let us now repeat our experiment with a symmetric string-coupled pendulum that has been set in motion by equal and opposite displacements along the Y-axis, but with S approaching zero:

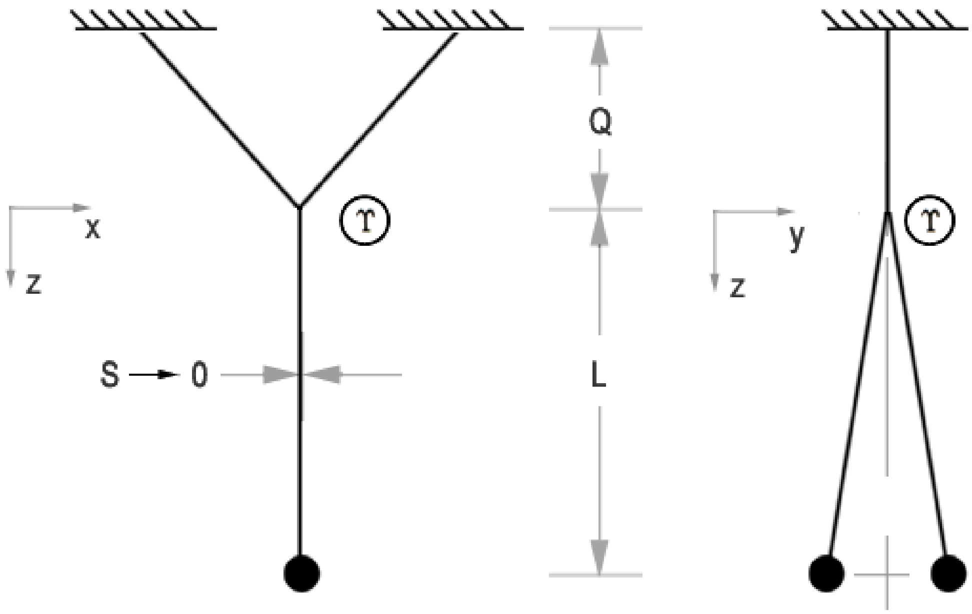


Fig. 3-4

Here again we observe the inner and outer cycles converging to a simple harmonic wave form, while the equally opposed forces developed through the symmetric movements of the two masses tend to peg node ϒ to a fixed point at a distance Q below the supports. In this case an analogous simple pendulum would have an equivalent length of L:

$$E = L \quad \text{for} \quad S \rightarrow 0$$ Eqa. 3-6

We may also observe that this equivalent length will be approached as Q approaches zero.

At this point we have nothing to lose by adding the assumption of a linear relationship between E and S to our growing list of what must be proven later. Postulating a linear linkage between Equations 3-5 and 3-6 suggests that the equivalent length E of a string-coupled pendulum, initialized by equal and opposite displacements along the Y-axis, to be that of a simple pendulum of length …

$$E = L + Q \cdot S / H$$

Eqa. 3-7

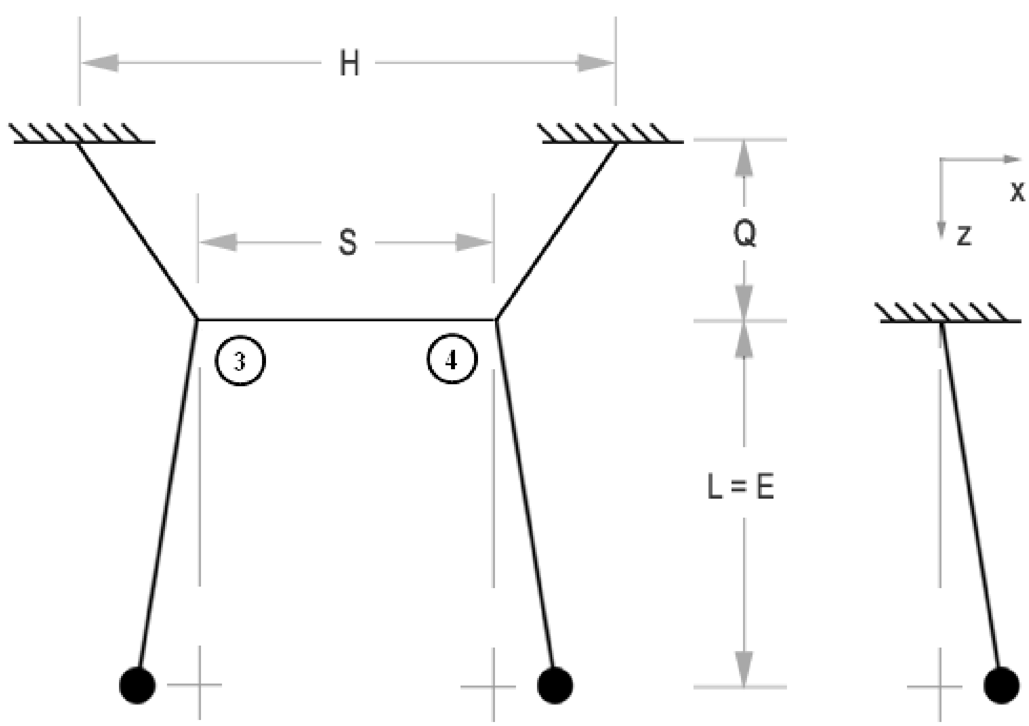
Recurring again to Equation 3-2, we now have an empirically derived opinion for the greater of the two frequencies governing motion in the Y-direction, viz.:

$$\omega_{1Y} = \sqrt{\frac{G}{L + Q \cdot S / H}} \qquad \text{Eqa. 3-8}$$

Article 4: Observations of motion along the X-axis. The two previous articles developed simple experiments for revealing the natural frequencies of motion along the Y-axis of the string-coupled pendulum. These methods can be applied directly to a similar derivation of the natural frequencies relating to motion in the X-direction.

We know that the larger of these two frequencies, ω_1 will correspond to the simple harmonic oscillations set in motion by equal and opposite initial displacements of the two pendulum weights along the X-axis, as shown in Figure 4-1:

Fig. 4-1



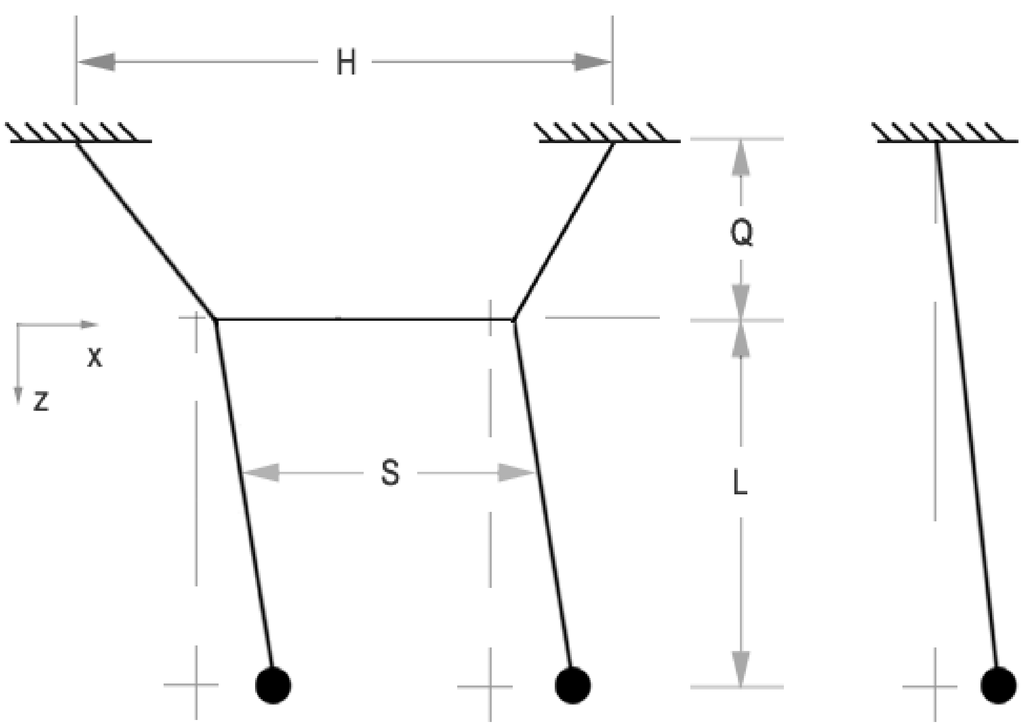
Here we observe that the equally opposed forces developed in the symmetric movements of the pendulum's masses will peg nodes ③ and ④ motionless in their rest positions, and that the only dimension entering into the dynamics of this system is L, hence:

$$\omega_{1X} = \sqrt{\frac{G}{L}} \qquad \text{Eqa. 4-1}$$

We would expect ω_{2X} to reveal itself when the initial X-displacements are alike in both magnitude and direction, as shown in Figure 4-2. In this instance we observe displacements of the entire plane of the string-coupled pendulum at a frequency corresponding to that of a simple pendulum with the same total distance between its point of support and the pendulum weight. The frequency of this motion is given by:

$$\omega_{2X} = \sqrt{\frac{G}{L+Q}} \qquad \text{Eqa. 4-2}$$

Fig. 4-2



Article 5: Pulses per half cycle in Y-motion. Let us now propose an empirical method for testing our experimental estimates for the internal frequencies, ω_1 and ω_2, governing bi-harmonic oscillations along both the X- and Y-axes of a string-coupled pendulum. Laboratory procedures for recording the instantaneous positions, velocities, and accelerations of moving bodies are far from trivial, requiring much more apparatus than would be in keeping with the spirit of using the straight-forward instrument of a simple pendulum to infer the behaviors of the string-coupled pendulum.

Returning to the equations for bi-harmonic motion that introduced Article 2, we can contrive another experiment to easily prove or disprove our work to this point. Please recall that these dynamics follow from an initial displacement of only one of the pendulum weights, while the other is left to begin movement from its rest position. We have postulated the following equation for the movement of one of the node points denoting the location of a pendulum weight:

$$P_1 = D \cdot \cos\left[\left(\frac{\omega_1 + \omega_2}{2}\right) \cdot t\right] \cdot \cos\left[\left(\frac{\omega_1 - \omega_2}{2}\right) \cdot t\right] \qquad \text{Eqa. 2-1}$$

And we have observed that frequencies ω_1 and, ω_2 in this equation define the periods of the inner and outer cycles of the bi-harmonic motion visualized in Figure 2-1.

These frequencies are apparent in an easily made observation of the number of times one of the pendulum weights will oscillate between those occasional cycles when it appears to be completely at rest. A count N of number of pulses per half cycle is given by half the ratio of larger period to the shorter period:

$$N = \frac{1}{2} \cdot \frac{\dfrac{2\pi}{(\omega_1 - \omega_2)/2}}{\dfrac{2\pi}{(\omega_1 + \omega_2)/2}} = \frac{1}{2} \cdot \frac{\omega_1 + \omega_2}{\omega_1 - \omega_2} \qquad \text{Eqa. 5-1}$$

The most pleasing development of pulses per half cycle available for Y-motion is rather indirect. It follows from establishing a common denominator for the internal frequencies ω_1 and ω_2. Equation 3-8 is modified in this direction by multiplying is numerator and denominator by $\sqrt{H(L+Q)}$:

$$\omega_{1Y} = \sqrt{\frac{G \cdot H \cdot (L + Q)}{(H \cdot L + S \cdot Q) \cdot (L + Q)}}$$ Eqa. 5-3

Carrying through with the indicated multiplication yields:

$$\omega_{1Y} = \sqrt{\frac{G \cdot H \cdot (L + Q)}{H \cdot L^2 + L \cdot Q \cdot S + H \cdot L \cdot Q + Q^2 S}}$$ Eqa. 5-4

A certain operation on the denominator in Equation 5-4 introduces an interesting possibility for recombining terms…

$$\omega_{1Y} = \sqrt{\frac{G \cdot H \cdot (L + Q)}{H \cdot L^2 + \frac{L \cdot Q \cdot S}{2} + \frac{L \cdot Q \cdot S}{2} + \frac{H \cdot L \cdot Q}{2} + \frac{H \cdot L \cdot Q}{2} + Q^2 S}}$$

Eqa. 5-5

$$\omega_{1Y} = \sqrt{\frac{G \cdot H \cdot (L + Q)}{H \cdot L^2 + \frac{L \cdot Q \cdot S}{2} + \frac{H \cdot L \cdot Q}{2} + Q^2 S + \frac{L \cdot Q \cdot S}{2} + \frac{H \cdot L \cdot Q}{2}}}$$

Eqa. 5-6

$$\omega_{1Y} = \sqrt{\frac{G \cdot H \cdot (L + Q)}{H \cdot L^2 + L \cdot Q \cdot \frac{H + S}{2} + Q^2 S + L \cdot Q \cdot \frac{H + S}{2}}}$$

Eqa. 5-7

resulting in:

$$\omega_{1Y} = \sqrt{\frac{G \cdot H \cdot (L + Q)}{L \cdot \left(H \cdot L + Q \cdot \frac{(H + S)}{2} \right) + Q \cdot \left(L \cdot \frac{(H + S)}{2} + QS \right)}}$$

Eqa. 5-8

Equation 5-8 presents us with easily measured areas in the X-Z plane …

$$\vartheta = H \cdot L + Q\frac{H+S}{2}$$ Eqa. 5-9

$$\varphi = Q\frac{H-S}{2}$$ Eqa. 5-10

$$\lambda = L\frac{H+S}{2} + Q \cdot S$$ Eqa. 5-11

that can be visualized in terms of Figure 5-1:

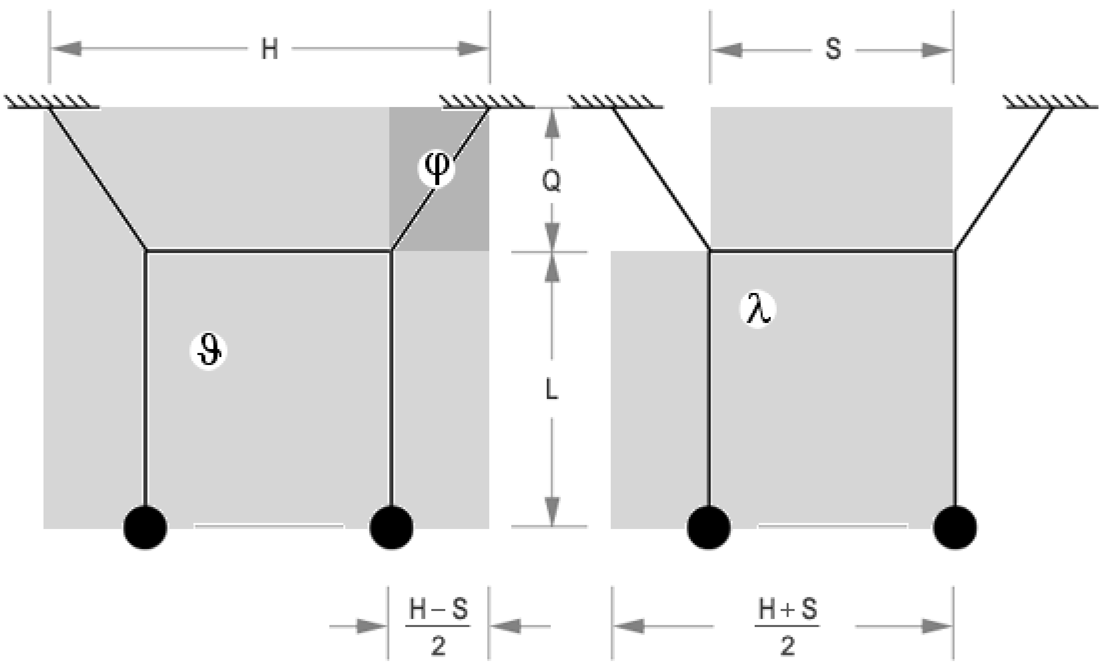


Fig. 5-1

These parameters allow ω_1's reformulation as …

$$\omega_{1Y} = \sqrt{\frac{G \cdot (\vartheta + \varphi)}{L \cdot \vartheta + Q \cdot \lambda}}$$ Eqa. 5-12

which creates attractive comparisons with ω_2 when Equation 3-3 is reformulated on the same common denominator …

$$\omega_{2Y} = \sqrt{\frac{G \cdot H \cdot (L + Q \cdot S/H)}{L \cdot \vartheta + Q \cdot \lambda}} \qquad \text{Eqa. 5-13}$$

namely:

$$\omega_{2Y} = \sqrt{\frac{G \cdot (\vartheta - \varphi)}{L \cdot \vartheta + Q \cdot \lambda}} \qquad \text{Eqa. 5-14}$$

Substituting Equations 5-12 and 5-14 into our Equation 5-1 for the number of pulses per half cycle yields:

$$N = \frac{1}{2} \cdot \frac{\sqrt{\vartheta + \varphi} + \sqrt{\vartheta - \varphi}}{\sqrt{\vartheta + \varphi} - \sqrt{\vartheta - \varphi}} \qquad \text{Eqa. 5-15}$$

Multiplying both the numerator and denominator of Equation 5-15 by the numerator will present a great many terms for elimination …

$$N = \frac{1}{2} \cdot \frac{\vartheta + \varphi + 2\sqrt{\vartheta + \varphi}\sqrt{\vartheta - \varphi} + \vartheta - \varphi}{\vartheta + \varphi - (\vartheta - \varphi)} \qquad \text{Eqa. 5-16}$$

leading to:

$$N = \frac{\vartheta + \sqrt{(\vartheta + \varphi) \cdot (\vartheta - \varphi)}}{2 \cdot \varphi} \qquad \text{Eqa. 5-17}$$

If we then note that $\varphi << \vartheta$ for pendula having the most attractive dynamics, we can finally arrive at a compact expression for Y-motion's number N of pulses per half cycle:

$$N \approx \frac{\vartheta}{\varphi} \qquad \text{Eqa. 5-18}$$

Article 6: Pulses per half cycle in X-motion. Developing the number of pulses per half cycle equation for X-motion is best accomplished through a more direct treatment of Equation 5-1. It begins by taking a ratio of the pendulum's two internal frequencies …

$$\omega = \frac{\omega_2}{\omega_1} \qquad \text{Eqa. 6-1}$$

which allows Equation 5-1 to be restated as:

$$N = \frac{1}{2} \cdot \frac{1+\omega}{1-\omega} \qquad \text{Eqa. 6-2}$$

For X-motion ω is given by references to Equations 4-1 and 4-2, which provide the appropriate ω_1 and ω_2 for entry into Equations 6-1 and 6-2. These substitutions …

$$\omega = \frac{\sqrt{\frac{G}{L+Q}}}{\sqrt{\frac{G}{L}}} = \sqrt{\frac{L}{L+Q}} \qquad \text{Eqa. 6-3}$$

yield the count N of the number of pulses per half cycle to be observed for a string-coupled pendulum exhibiting bi-harmonic motion along its X-axis:

$$N = \frac{1}{2} \cdot \frac{1+\sqrt{\frac{L}{L+Q}}}{1-\sqrt{\frac{L}{L+Q}}} \qquad \text{Eqa. 6-4}$$

Placing this equation's right-most ratio on a common denominator …

$$N = \frac{1}{2} \cdot \frac{\sqrt{L+Q}+\sqrt{L}}{\sqrt{L+Q}-\sqrt{L}} \qquad \text{Eqa. 6-5}$$

yields an opportunity for simplification by multiplying the numerator and the denominator by the numerator:

$$N = \frac{1}{2} \cdot \frac{L + Q + 2\sqrt{L}\sqrt{L + Q} + L}{L + Q - L}$$ Eqa. 6-6

Carrying-out the operations indicated brings us to …

$$N = \frac{L}{Q} + \frac{1}{2} + \sqrt{\frac{L^2 + LQ}{Q^2}}$$ Eqa. 6-7

which ultimately reveals the primacy of a ratio L:Q in governing X-motion:

$$N = \frac{1}{2} + \frac{L}{Q} + \sqrt{\frac{L}{Q}\left(\frac{L}{Q} + 1\right)}$$ Eqa. 6-8

Thus we observe that the dynamics of X-motion are entirely controlled by dimensions in the Z-direction, while H and S are irrelevant. Noting that the string-coupled pendulum's horizontal dimensions do not enter into the dynamics of motion taking place entirely along the X-axis provides some important criteria for constructing the principles to be used in our formal development of the parameters and equations governing string-coupled pendula:

> Since Y-motion does not distort the figure that the pendulum presents to the X-Z plane, it cannot cause the X-components of reaction to vary from their rest values.
>
> X-motion, on the other hand, can only be governed by the impulses arising from the differences between these reactions.

Thus our final observation here is that the dimensions H and S are significant to the string-coupled pendulum only in that they specify the rest values of the X-components of the support reactions that somehow enter into the values taken on by the Y-reactions during movements along the Y-axis. The H and S dimensions do not enter into purely X-motion because the rest values of the X-reactions are equal and opposite and will, therefore, cancel out when the net impulse from these reactions is determined.

Article 7: Principles of analysis. Our formal derivation of the equations and parameters governing the movements of string-coupled pendula will be based on extensions of the familiar principles commonly used in deriving the simple pendulum's equation of motion. The elementary free body diagram of Figure 7-1 conveys a sense of the flow of forces in a simple pendulum.

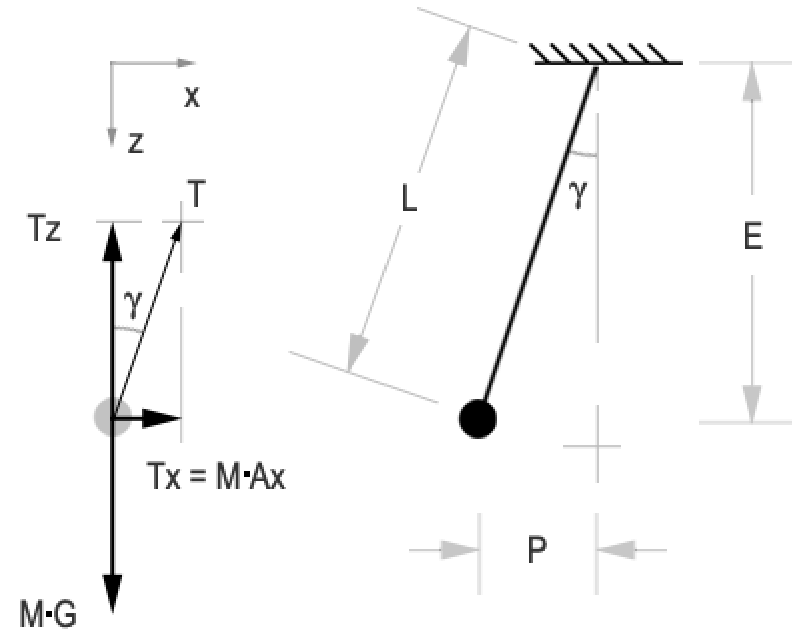


Fig. 7-1

In this figure a simple pendulum of length L supports a weight of mass M in a gravitational field of intensity G. Displacements from rest along the X-axis are measured by the position variable P; accelerations in this direction are denoted by A, which is the second derivative of P with respect to time; and the components of the tension in the string, T, resolve into T_z and T_x along the coordinate axes.

Our basic principle for analyzing this figure is simply the definition of a string as it is used in structural analysis, viz.: an idealized string offers no resistance to forces that would tend to bend it. Observing that the string joining the pendulum weight to its support remains straight during oscillations, we must assume that all the forces on the pendulum weight always resolve into the axis of the string. The foregoing can be summarized in saying that the angle γ at which the pendulum is inclined must coincide with the inclinations of the tension vector T at all times. Mathematically, this condition can be expressed as:

$$\frac{T_X}{T_Z} = \frac{P}{E} \qquad \text{Eqa. 7-1}$$

The force diagram in Figure 7-1 allows us to read off the following equations for the components of tension:

$$T_X = M \cdot A \qquad \text{Eqa. 7-2}$$

$$T_Z = -M \cdot G \qquad \text{Eqa. 7-3}$$

Substituting these components of tension into Equation 7-1 and re-arranging will yield:

$$P = -\frac{E}{G} \cdot A \qquad \text{Eqa. 7-4}$$

Since A is defined as the second time-derivative of P this would be a fairly tractable differential equation until we observe that E is related to P by way of Pythagoras:

$$E^2 = L^2 - P^2 \qquad \text{Eqa. 7-5}$$

At this point, the typical procedure is to note that $\sqrt{(L^2 + P^2)}$ is not greatly different from L when P is small In comparison to L. With this provision, we can substitute L for E in Equation 7-4 to arrive at Equation 7-6 for the motion of a simple pendulum:

$$P = -A \cdot G/L \qquad \text{Eqa. 7-6}$$

The equation above defines the simple harmonic motion that is generally thought of as being epitomized by simple pendula. The behavior modes to which the variable P is constrained by Equation 7-6 can be shown via direct substitution to be fully expressed in Equation 7-7 …

$$P(t) = D \cdot \cos(\omega \cdot t + \delta) \qquad \text{Eqa. 7-7}$$

where D arises from an initial displacement of the pendulum weight along the X-axis and ω is given by $\sqrt{G/L}$. The parameter δ allows for motions that are initiated by imparting an initial velocity to the pendulum weight. Having mentioned the possibility of motions originated in this manner, we are going to drop the parameter δ from consideration at this point. Our feeling is that the motions of a string-coupled pendulum can be made fully apparent through initiating stimuli that are entirely confined to displacements of the two pendulum weights at time t = 0; and that an exhaustive consideration of all the possible initial conditions would quadruple the number of equations and parameters that would have to be treated, while adding nothing to the entertainment value of our presentation. Thus our references to the equation for the motion of a simple pendulum will be:

$$P(t) = D \cdot \cos(\omega \cdot t) \qquad \text{Eqa. 7-8}$$

A passing observation may be made to the effect that our treatment of the simple pendulum allows no deviations in the vertical component of the support reaction from its rest value of M·G. Since our analyses have lost sight of vertical displacements of the pendulum weight when E was equated to L in Equation 7-6, there are no vertical accelerations to justify any unbalanced forces along this axis. Ostensibly the deviations of the vertical support reaction would be relatively small, just as P is small relative to L, so that implications of our observation have little significance for the simple pendulum. But, when we transfer our understanding of the simple pendulum to the analysis of string-coupled pendula, the simplifications which follow from the assumption of zero vertical displacements become significant in a number of ways:

> First it should be noted that if nodes ① and ② are not displaced vertically by either X- or Y-motion, then nodes ③ and ④ will not be displaced vertically either. This means that the distance between nodes ③ and ④ does not change, and that the X-components for the positions of these nodes must therefore differ by the distance S at all times.
>
> A second manifestation of our assumption of zero vertical displacements is that all motions in the X-Y plane can be decomposed along the X- and Y-axes in a manner showing the movements along one axis to be independent of the movements taking place along the other axis. On first consideration this notion might appear to be arguable from the standpoint of a balance of moments in the X-Y plane, where it is clear that the accelerations and displacement along the two axes do interact to specify the Y-components of reaction in ways that differ according to differing degrees of the displacements. There are a number of ways to approach a point of view in which these interactions appear to be of the ‘second-order’ variety that were dismissed in our analysis of simple pendula:
>
>> Our initial observation might be to the effect that all variations in the moment arms owing to displacements in the X-Y plane are small in comparison to the dimensions H and S, and that the forces developed in this plane are small in comparison the constant pull of gravity on the pendulum weights. This approach is essentially a translation of the simplifying assumptions that were made for a simple pendulum to a horizontal plane.
>>
>> Another approach would originate in the observation that assuming zero vertical displacements confers the identity of a linear, elastic system on the string-coupled pendulum: in displacing a pendulum weight in the horizontal plane, one encounters a resisting force that is proportional to the amount of the displacement (see Equation 7-1). Hence we should again expect the principle of structural

superposition to apply, which it does in that all changes in the length of moment arms in the X-Y plane will be exactly off-set by an opposing variation in the force acting through that moment arm.

A third manifestation of the assumption of zero vertical displacements for a string-coupled pendulum is that the vertical components of reaction cannot change as a result of the motions taking place in the X-Y plane. Here we approach, by another path, the conclusion drawn from Article 6's investigation of pulses per half-cycle in exclusively X-motion:

Since Y-motion does not distort the figure that the pendulum presents to the X-Z plane, it cannot cause the horizontal components of reaction to vary from their rest values.

X-motion, on the other hand, can only be governed by the impulses arising from the differences between these reactions.

These notions should be problematic only for X-motion, since movements along the Y-axis can be discounted in their effect on vertical reaction components on the basis of the same reasoning that was presented for a simple pendulum. But it is not entirely clear that the displacements taking place in the X-Z plane have been constrained so that they will always conspire with the corresponding array of forces in a manner that keeps the burden of supporting the pendulum weights equally distributed between the two reaction points.

Our formal development of the equations of motion for string-coupled pendula will consider the X- and Y-components of motion separately, and then assert that the principle of structural superposition allows us to consider that these equations hold regardless of what may be happening along a mutually orthogonal axis.

The preparation for such an assertion would be entirely complete at this point except that our development of the equations for Y-motion depends on constant vertical reactions. Thus the discerning reader may not allow that the equations for Y-motion hold when X-motion is present. In order to avoid the possibility of appearing to use the principle of structural superposition in a facile or merely circular manner, let us accept the obligation to demonstrate the constant nature of the vertical reactions as part of the development of our equations for X-motion.

Article 8: The equations of motion. There are essentially three considerations governing the forms of movement that a string-coupled pendulum might demonstrate. The first of these are the ‘pendulum’ equations analogous to Equation 7-6. As shown in Figure 8-1, these equations require a reference to the positions of nodes ③ and ④ in order to relate the accelerations of nodes ① and ② to the inclinations to the strings joining the respective nodes:

$$A_{1X} = -G \cdot (P_{1X} - P_{3X})/L \qquad \text{Eqa. 8-1}$$

$$A_{2X} = -G \cdot (P_{2X} - P_{4X})/L \qquad \text{Eqa. 8-2}$$

$$A_{1Y} = -G \cdot (P_{1Y} - P_{3Y})/L \qquad \text{Eqa. 8-3}$$

$$A_{2Y} = -G \cdot (P_{2Y} - P_{4Y})/L \qquad \text{Eqa. 8-4}$$

The structural definition of a string imposes a similar set of constraints on the relationships between support reactions and the inclinations of the strings at the points of support. A mathematical expression of this notion would requires that forces acting at nodes ⑤ and ⑥ must resolve into the axis of the string connected to that node:

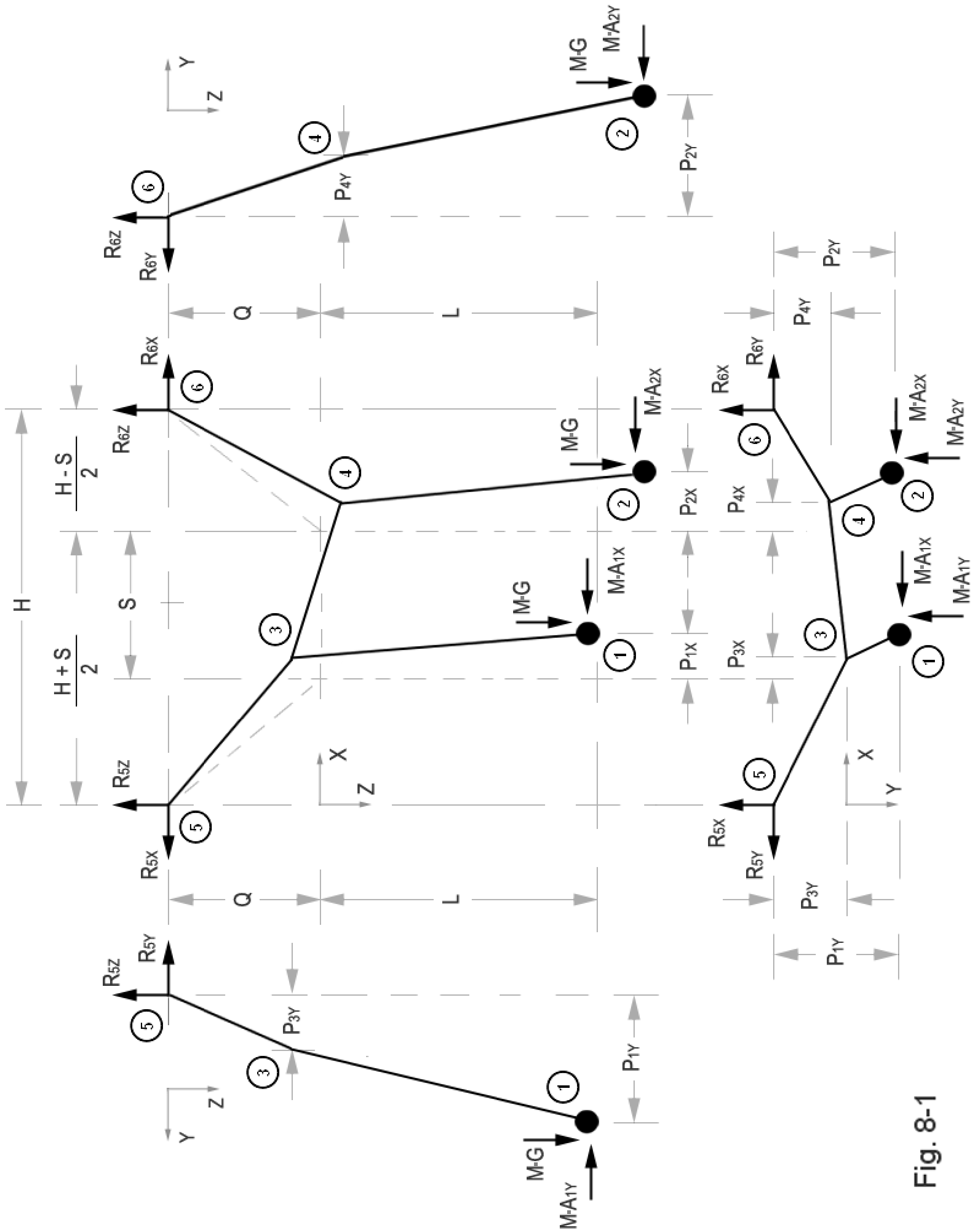
$$R_{5X}/R_{5Z} = \left[\frac{(H-S)}{2} + P_{3X}\right]\Big/ Q \qquad \text{Eqa. 8-5}$$

$$R_{6X}/R_{6Z} = \left[\frac{(H-S)}{2} - P_{4X}\right]\Big/ Q \qquad \text{Eqa. 8-6}$$

$$R_{5Y}/R_{5Z} = P_{3Y}/Q \qquad \text{Eqa. 8-7}$$

$$R_{6Y}/R_{6Z} = P_{4Y}/Q \qquad \text{Eqa. 8-8}$$

A third group of equations arises from the definition of structural support, which can be said to generate just that amount of force necessary to keep the support point stationary at all times. These equations state 1) an overall balance of the forces acting along each coordinate axis, and 2) requirements that the resultant of all moments in each plane of motion be zero.



Y
Z
X
M·G
M·A2Y
M·A2X
M·A1X
M·A1Y
R6Z
R6Y
R6X
R5Z
R5Y
R5X
P4Y
P2Y
P3Y
P1Y
P2X
P4X
P1X
P3X
Q
L
H
S
H - S
2
H + S
2
1
2
3
4
5
6

Fig. 8-1

The equation assuring that nodes ⑤ and ⑥ do not rotate in the Y-Z plane expresses a balance of moments about node ⑤:

$$\sum M_{⑤} \circlearrowleft^{+} = 0 \qquad 0 = -M \cdot (A_{1Y} + A_{2Y}) \cdot (L + Q) - M \cdot G \cdot (P_{1Y} + P_{2Y}) \qquad \text{Eqa. 8-9}$$

The same consideration in the X-Z plane requires that:

$$\sum M_{⑤} \circlearrowleft^{+} = 0 \qquad 0 = R_{6Z} \cdot H - M \cdot (A_{1X} + A_{2X}) \cdot (L + Q) - M \cdot G \cdot \left(\frac{H+S}{2} + P_{2X} + \frac{H-S}{2} + P_{1X} \right) \qquad \text{Eqa. 8-10}$$

If the supports are to remain stationary in relation to the Y-axis, the reaction components must obey Equation 8-11:

$$\sum F_{Y} \overset{+}{\Rightarrow} = 0 \qquad 0 = R_{5Y} + R_{6Y} + M \cdot (A_{1Y} + A_{2Y}) \qquad \text{Eqa. 8-11}$$

A similar equation is required to fix the supports along the X-axis:

$$\sum F_{X} \overset{+}{\Rightarrow} = 0 \qquad 0 = R_{5X} - R_{6X} + M \cdot (A_{1X} + A_{2X}) \qquad \text{Eqa. 8-12}$$

At this point we have established twelve equations and introduced fourteen variables, viz.: six components of reaction, R_{5X}, R_{5Y}, R_{5Z}, R_{6X}, R_{6Y}, and R_{6Z}; four variables specifying the positions of the intermediate node points, P_{3X}, P_{4X}, P_{3Y}, and P_{4Y}; and four variables specifying the positions of the pendulum weights, P_{1X}, P_{2X}, P_{1Y}, and P_{2Y}. (Recall that all Z-displacements have been disregarded.)

These equations have been chosen so as to define a system that we anticipate will decompose into independent subsystems for governing movements along the two axes of motion. As shown in Table 8-1, both halves of

Unknowns	X-motion	Y-motion
1.	R_{5Z}	R_{5Z}
2.	R_{6Z}	R_{6Z}
3.	R_{5X}	R_{5Y}
4.	R_{6X}	R_{6Y}
5.	P_{3X}	P_{3Y}
6.	P_{4X}	P_{4Y}
7.	P_{1X}	P_{1Y}
8.	P_{2X}	P_{2Y}

Tbl. 8-1

this system involve eight variables, with the vertical components of reaction, R_{5Z} and R_{6Z}, being common to each subsystem. (This commonality expresses concerns regarding the independence of X- and Y-motions stated earlier.)

We can begin to make up our shortage of equations by a statement of vertical equilibrium …

$$R_{5Z} + R_{6Z} = 2 \cdot M \cdot G \qquad \text{Eqa. 8-13}$$

that can be applied to both X- and Y-motions. One additional equation is also available from the consideration of external equilibrium, viz.: the balance of moments in the X-Y plane:

$$\sum M_{⑤} = 0 \qquad 0 = R_{6Y} \cdot H + M \cdot A_{1Y} \cdot \left(\frac{H-S}{2} + P_{1X} \right) + M \cdot A_{2Y} \cdot \left(\frac{H+S}{2} + P_{2X} \right) - M \cdot \left(A_{1X} \cdot P_{1Y} + A_{2X} \cdot P_{2Y} \right) \qquad \text{Eqa. 8-14}$$

But we must note that of the six such equations requiring stationary supports (8-5 through 8-8, 8-13, and 8-14) any five are sufficient to specify the sixth. The remaining equations required to fully specify X- and Y-motion must therefore arise from our earlier consideration of analytical principles. To completely specify movements along the X-axis we introduce our requirement that the X-distance between the intermediate nodes must remain constant:

$$P_{3X} = P_{4X} \qquad \text{Eqa. 8-15}$$

Having established that our approximate view of the string-coupled pendulum has displacements in the Y-direction doing nothing to change the vertical reactions …

$$R_{5Z} = R_{6Z} (= M \cdot G) \qquad \text{Eqa. 8-16}$$

we have also introduced the suspicion that X-movements will leave this equation unchanged. But, having introduced the vertical reactions as variables in the system of equations specifying X-motion, we are now in a position to allow this equation to manifest itself as being valid through all the behavior modes of a string-coupled pendulum.

Article 9: X-motion. A rigorous general solution to our non-linear system of eight equations in eight unknowns can be successfully approached using a number of schemes for substitutions and combinations of variables. Recalling some of the least incisive of these, our essay might have been a hundred pages longer had we not discovered that the pendulum weights always move in simple harmonic manner *relative to one another* when considered from the standpoint of movements along a single coordinate axis. Let us introduce the following change of notation to demonstrate and take advantage of this regularity:

$$a_1 = (A_{1X} + A_{2X}) \qquad \text{Eqa. 9-1}$$

$$a_2 = (A_{1X} - A_{2X}) \qquad \text{Eqa. 9-2}$$

$$p_1 = (P_{1X} + P_{2X}) \qquad \text{Eqa. 9-3}$$

$$p_2 = (P_{1X} - P_{2X}) \qquad \text{Eqa. 9-4}$$

Re-stating the pendulum Equations 8-1 and 8-2 in terms of our new variables, and invoking the equality between P_{3X} and P_{4X}, yields one simple harmonic equation …

$$a_2 = -(G/L) \cdot p_2 \qquad \text{Eqa. 9-5}$$

and another that remains a bit complicated:

$$a_1 = -(G/L) \cdot (p_1 - P_{3X} - P_{4X}) \qquad \text{Eqa. 9-6}$$

Equation 8-10 can also be simplified by our change in variables …

$$R_{6Z} \cdot H = a_1 \cdot M \cdot (L + Q) + M \cdot G \cdot (H + p_1) \qquad \text{Eqa. 9-7}$$

as can Equation 8-12:

$$R_{6X} = R_{5X} + M \cdot a_1 \qquad \text{Eqa. 9-8}$$

Equations 8-5 and 8-6 are brought into the analysis by solving for P_{3X} and P_{4X}:

$$P_{3X} = \frac{Q \cdot R_{5X}}{R_{5Z}} - \frac{H - S}{2}$$ Eqa. 9-9

$$P_{4X} = \frac{-Q \cdot R_{6X}}{R_{6Z}} + \frac{H - S}{2}$$ Eqa. 9-10

Adding Equations 9-9 and 9-10 yields:

$$P_{3X} + P_{4X} = Q \cdot \left(\frac{R_{5X}}{R_{5Z}} - \frac{R_{6X}}{R_{6Z}} \right)$$ Eqa. 9-11

This equation can be re-stated entirely in terms of the reactions at node ⑥ by using Equation 9-8 to eliminate R_{5X} and Equation 8-13 to eliminate R_{5Z}:

$$P_{3X} + P_{4X} = Q \cdot \left(\frac{R_{6X} - M \cdot a_1}{2 \cdot M \cdot G - R_{6Z}} - \frac{R_{6X}}{R_{6Z}} \right)$$ Eqa. 9-12

Placing Equation 9-12's right-most terms on a common denominator prepares for an operation that will eliminate R_{6X}:

$$P_{3X} + P_{4X} = \frac{Q \cdot R_{6X} - Q \cdot M \cdot a_1}{2 \cdot M \cdot G - R_{6Z}} - \frac{Q \cdot R_{6X}}{R_{6Z}}$$ Eqa. 9-13

$$P_{3X} + P_{4X} = \frac{-R_{6Z} \cdot Q \cdot R_{6X} + R_{6Z} \cdot Q \cdot M \cdot a_1}{-R_{6Z} \cdot (2 \cdot M \cdot G - R_{6Z})}$$ Eqa. 9-14

$$+ \frac{Q \cdot R_{6X} \cdot (2 \cdot M \cdot G - R_{6Z})}{-R_{6Z} \cdot (2 \cdot M \cdot G - R_{6Z})}$$

$$P_{3X} + P_{4X} = \frac{-R_{6Z} \cdot Q \cdot R_{6X} + R_{6Z} \cdot Q \cdot M \cdot a_1 + 2 \cdot Q \cdot M \cdot G \cdot R_{6X} - R_{6Z} \cdot Q \cdot R_{6X}}{-R_{6Z} \cdot (2 \cdot M \cdot G - R_{6Z})}$$

Eqa. 9-15

$$P_{3X} + P_{4X} = \frac{(-2 \cdot R_{6Z} + 2 \cdot M \cdot G) \cdot Q \cdot R_{6X} + R_{6Z} \cdot Q \cdot M \cdot a_1}{-R_{6Z} \cdot (2 \cdot M \cdot G - R_{6Z})}$$

Eqa. 9-16

Substituting this rearrangement of Equation 9-10 …

$$Q \cdot R_{6X} = -R_{6Z}\left(P_{4X} - \frac{H-S}{2}\right)$$

Eqa. 9-17

into Equation 9-16 …

$$P_{3X} + P_{4X} = \frac{(R_{6Z} - M \cdot G) \cdot (H - S - 2 \cdot P_{4X}) - M \cdot Q \cdot a_1}{2 \cdot M \cdot G - R_{6Z}}$$

Eqa. 9-18

accomplishes the elimination of R_{6X}. And recalling from Equation 8-15 that P_{4X} can be substituted for P_{3X} also eliminates P_{3X} from the above:

$$2 \cdot P_{4X} = \frac{-2 \cdot P_{4X} \cdot (R_{6Z} - M \cdot G) + (H - S) \cdot (R_{6Z} - M \cdot G) - M \cdot Q \cdot a_1}{2 \cdot M \cdot G - R_{6Z}}$$

Eqa. 9-19

Further reductions on Equation 9-19 serve to isolate the variables P_{4X} and R_{6Z}:

$$2 \cdot P_{4X} \cdot (2 \cdot M \cdot G - R_{6Z}) = -2 \cdot P_{4X} \cdot (R_{6Z} - M \cdot G) + (H - S) \cdot (R_{6Z} - M \cdot G) - M \cdot Q \cdot a_1$$

Eqa. 9-20

$$4 \cdot P_{4X} \cdot M \cdot G = 2 \cdot P_{4X} \cdot M \cdot G + (H - S) \cdot (R_{6Z} - M \cdot G) - M \cdot Q \cdot a_1$$

Eqa. 9-21

$$2 \cdot P_{4X} = \frac{(H - S) \cdot (R_{6Z} - M \cdot G)}{M \cdot G} - \frac{Q \cdot a_1}{G}$$

Eqa. 9-22

Once again noting Equation 8-15's assertion that P_{3x} and P_{4x} are identical, Equation 9-6 can be rearranged to …

$$2 \cdot P_{4X} = \frac{L}{G} a_1 + p_1 \qquad \text{Eqa. 9-23}$$

which allows us to eliminate P_{4x} by combining Equations 9-22 and 9-23:

$$\frac{L}{G} \cdot a_1 + p_1 = \frac{(H - S) \cdot (R_{6Z} - M \cdot G)}{M \cdot G} - \frac{Q \cdot a_1}{G} \qquad \text{Eqa. 9-24}$$

Elimination of the reaction R_{6z} from Equation 9-24 requires a rearrangement of Equation 9-7:

$$R_{6Z} - M \cdot G = \frac{a_1 \cdot M \cdot (L + Q) + M \cdot G \cdot p_1}{H} \qquad \text{Eqa. 9-25}$$

Combining Equations 9-24 and 9-25 leads to …

$$\frac{L}{G} \cdot a_1 + p_1 = \frac{(1 - S/H) \cdot a_1 \cdot (L + Q)}{G} - \frac{Q \cdot a_1}{G} \qquad \text{Eqa. 9-26}$$

which is most conveniently arranged as :

$$a_1 \cdot (L + Q)/G + p_1 = (1 - S/H) \cdot (a_1 \cdot (L + Q)/G + p_1) \qquad \text{Eqa. 9-27}$$

Equation 9-27 has two solutions that can be immediately noted and disregarded: S = H produces simple harmonic motion described in Equation 4-2 and visualized in Figure 4-2; and S = 0 reduces this equation to an identity. One solution to Equation 9-27 preserves the time-variant behaviors we seek to explain in an equation for simple harmonic (relative) motion, viz.:

$$a_1 = -G \cdot p_1/(L + Q) \qquad \text{Eqa. 9-28}$$

While this equation is at hand let us use it to substitute for the a_1 of Equation 9-7 in order to establish the behavior of the vertical reactions during displacements along the X-axis:

$$R_{6Z} \cdot H = \frac{-G \cdot p_1}{(L+Q)} \cdot M \cdot (L+Q) + M \cdot G \cdot (H + p_1) = M \cdot G \cdot H$$

Eqa. 9-29

Here we have a validation of our postulate that each vertical reaction retains a constant value of M•G even while the pendulum weights are displaced along the X-axis.

Now let us return to a consideration of Equations 9-5 and 9-28, which simultaneously specify the positions of the pendulum weights along the X-axis. Continuing with our practice of disregarding motions that are initiated by any means except in initial displacement of the pendulum weights, we can solve these two equations by inspection:

$$p_1 = P_{1X} + P_{2X} = C_{1X} \cdot \cos(\omega_{2X} \cdot t)$$

Eqa. 9-30

$$p_2 = P_{1X} - P_{2X} = C_{2X} \cdot \cos(\omega_{1X} \cdot t)$$

Eqa. 9-31

where C_{1X} and C_{2X} are the initial relative displacements of the two pendulum weights, and ω_1 and ω_2 have the same identities they were given in Article 4, viz.:

$$\omega_{1X} = \sqrt{\frac{G}{L}}$$

Eqa. 4-1

$$\omega_{2X} = \sqrt{\frac{G}{L+Q}}$$

Eqa. 4-2

Equations 9-30 and 9-31 can be solved together to yield separate equations for the positions of the pendulum weights throughout time:

$$2 \cdot P_{1X} = C_{1X} \cdot \cos(\omega_{2X} \cdot t) + C_{2X} \cdot \cos(\omega_{1X} \cdot t)$$

Eqa. 9-32

$$2 \cdot P_{2X} = C_{1X} \cdot \cos(\omega_{2X} \cdot t) - C_{2X} \cdot \cos(\omega_{1X} \cdot t)$$

Eqa. 9-33

In order to link these equations to the analysis proceeding from Equations 2-1 and 2-2, let us evaluate the above in terms of the initial conditions giving rise to the observations that were made in Article 2, viz.: at time equals zero, P_1 was equal to D and P_2 was zero. Combining these considerations with Equations 9-32 and 9-33 will specify $C_{1X} = C_{2X} = D$:

$$P_{1X} = \frac{D}{2} \cdot \left[\cos(\omega_{2X} \cdot t) + \cos(\omega_{1X} \cdot t)\right] \qquad \text{Eqa. 9-34}$$

$$P_{2X} = \frac{D}{2} \cdot \left[\cos(\omega_{2X} \cdot t) - \cos(\omega_{1X} \cdot t)\right] \qquad \text{Eqa. 9-35}$$

Using the trigonometric identities for the sums and differences of angles that were introduced in Article 2, we can transform each of the above into the standard form for bi-harmonic motion …

$$P_{1X} = D \cdot \cos\left(\frac{\omega_{1X} + \omega_{2X}}{2} \cdot t\right) \cdot \cos\left(\frac{\omega_{1X} - \omega_{2X}}{2} \cdot t\right) \qquad \text{Eqa. 9-36}$$

$$P_{2X} = D \cdot \sin\left(\frac{\omega_{1X} + \omega_{2X}}{2} \cdot t\right) \cdot \sin\left(\frac{\omega_{1X} - \omega_{2X}}{2} \cdot t\right) \qquad \text{Eqa. 9-37}$$

where D is the initial displacement of the first pendulum weight, and the second pendulum weight starts from rest.

This article has now validated the speculative discourse of Articles 2 and 4 on the basis of the laws of motion and our principles of analysis. We have also established that our principles of analysis imply vertical reactions that are not altered by displacements in the planes of motion, and that our mathematical approximation of a string-coupled pendulum is therefore linear and elastic in regard to both X- and Y-motion. Hence we may now claim that our principles of analysis contain the sanction for our linear superposition of the effects that the motion of one pendulum weight has on the other, and for our decomposition of all motions into X- and Y-components.

Article 10: Y-motion. From our experience in the previous article, we should expect that a change of variables emphasizing the displacements of the pendulum weights relative to one another will greatly simplify our analysis for motion along the Y-axis.

$$a_1 = (A_{1Y} + A_{2Y}) \qquad \text{Eqa. 10-1}$$

$$a_2 = (A_{1Y} - A_{2Y}) \qquad \text{Eqa. 10-2}$$

$$p_1 = (P_{1Y} + P_{2Y}) \qquad \text{Eqa. 10-3}$$

$$p_2 = (P_{1Y} - P_{2Y}) \qquad \text{Eqa. 10-4}$$

Restating the pendulum Equations 8-3 and 8-4 for Y-motion in terms of these new variables yields:

$$a_1 = -(G/L) \cdot (p_1 - P_{3Y} - P_{4Y}) \qquad \text{Eqa. 10-5}$$

$$a_2 = -(G/L) \cdot (p_2 - P_{3Y} + P_{4Y}) \qquad \text{Eqa. 10-6}$$

A simple harmonic relationship will appear immediately upon re-stating Equation 8-9 in terms of these new variables:

$$a_1 = -[G/(L + Q)] \cdot p_1 \qquad \text{Eqa. 10-7}$$

Equation 8-11 is also greatly simplified by this procedure:

$$a_1 = -(R_{5Y} + R_{6Y})/M \qquad \text{Eqa. 10-8}$$

Equations 8-7 and 8-8 express the relationships between reactions and the intermediate node points:

$$P_{3Y} = Q \cdot R_{5Y}/R_{5Z} \qquad \text{Eqa. 10-9}$$

$$P_{4Y} = Q \cdot R_{6Y}/R_{6Z} \qquad \text{Eqa. 10-10}$$

It is unfortunate that this particular compliment of equations does not contain enough information to specify the variable a_2, while a_1 is completely specified by Equation 10-9, and Equation 10-10 is merely redundant. Clearly one of these equations must be dropped in favor of a transformation upon Equation 8-14:

$$-R_{6Y}\cdot H = M\cdot A_{1Y}\frac{(H-S)}{2} + M\cdot A_{2Y}\frac{(H+S)}{2} \qquad \text{Eqa. 10-11}$$

Here we have ignored moments arising from displacements of the pendulum weights along the X-axis in accordance with the principles of analysis that imply a complete separation between actions taking place along the axes of motion. (Please recall from Article 7 that the practical implications of ignoring X-displacements in the moment equation for the X-Y plane was that moments arising from accelerations in the X-direction result in off-setting moments which arise from changing the length of the moment arms through which the Y-forces are acting.)

Re-expressing Equation 10-11 in terms our change in variables proceeds thusly:

$$R_{6Y} = -\frac{M}{2}\left[A_{1Y}\left(1-S/H\right) + A_{2Y}\left(1+S/H\right)\right] \qquad \text{Eqa. 10-12}$$

$$R_{6Y} = -\frac{M}{2}\left[A_{1Y} + A_{2Y} - (A_{1Y} - A_{2Y})S/H\right] \qquad \text{Eqa. 10-13}$$

$$R_{6Y} = -\frac{M}{2}\cdot\left(a_1 - S\cdot\frac{a_2}{H}\right) \qquad \text{Eqa. 10-14}$$

The reaction forces can be eliminated from our development by combining Equations 10-8, 10-9, 10-10, and 8-16. The reactions are brought together by adding Equations 10-9 and 10-10:

$$P_{3Y} + P_{4Y} = Q\cdot\left(\frac{R_{5Y}}{R_{5Z}} + \frac{R_{6Y}}{R_{6Z}}\right) \qquad \text{Eqa. 10-15}$$

Equation 8-16 sanctions replacing Equation 10-15’s vertical reactions with the constant M·G:

$$P_{3Y} + P_{4Y} = \frac{Q}{M\cdot G}\cdot\left(R_{5Y} + R_{6Y}\right) \qquad \text{Eqa. 10-16}$$

And Equation 10-8 then allows the replacement of $R_{5Y} + R_{6Y}$ by $-M \cdot a_1$:

$$P_{3Y} + P_{4Y} = -\frac{Q}{G} \cdot a_1 \qquad \text{Eqa. 10-17}$$

Equation 10-5 can also be solved for $P_{3Y} + P_{4Y}$:

$$P_{3Y} + P_{4Y} = \frac{L}{G} \cdot a_1 + p_1 \qquad \text{Eqa. 10-18}$$

Equating the right-hand sides of Equations 10-17 and 10-18 we arrive at …

$$a_1 = -\frac{G}{L} \cdot p_1 - \frac{Q}{L} \cdot a_1 \qquad \text{Eqa. 10-19}$$

which can be re-arranged in the exact form of Equation 10-7, thereby admitting this equation back into our system for Y-motion while allowing us to drop the consideration of moments in the Y-Z plane that had produced this equation by the direct observation recorded in Equation 8-9.

Having thus reconstituted our system of equations for Y-motion we can now use Equation 10-11 to isolate the variable a_2. Proceeding as above, we begin by subtracting Equation 10-9 from Equation 10-10 …

$$P_{4Y} - P_{3Y} = Q \cdot \left(\frac{R_{6Y}}{R_{6Z}} - \frac{R_{5Y}}{R_{5Z}} \right) \qquad \text{Eqa. 10-20}$$

and once again rely on Equation 8-16 to sanction the replacement of Equation 10-20's vertical reactions with the constant $M \cdot G$:

$$M \cdot G \cdot \frac{(P_{4Y} - P_{3Y})}{Q} = R_{6Y} - R_{5Y} \qquad \text{Eqa. 10-21}$$

Equation 10-8 can be rearranged …

$$R_{5Y} = R_{6Y} - M \cdot a_1 \qquad \text{Eqa. 10-22}$$

to allow a substitution for R_{5Y} in Equation 10-21:

$$M \cdot G \cdot \frac{(P_{4Y} - P_{3Y})}{Q} = 2 \cdot R_{6Y} + M \cdot a_1$$ Eqa. 10-23

Equation 10-14 can be rearranged …

$$2 \cdot R_{6Y} + M \cdot a_1 = M \cdot a_2 \cdot \frac{S}{H}$$ Eqa. 10-24

to allow a combination of Equations 10-23 with 10-24 that eliminates both R_{6Y} and the variable a_1:

$$P_{4Y} - P_{3Y} = \frac{Q}{G} \cdot \frac{S}{H} \cdot a_2$$ Eqa. 10-25

Rearranging Equation 10-6 to isolate the intermediate nodes …

$$P_{4Y} - P_{3Y} = -\frac{L}{G} \cdot a_2 - p_1$$ Eqa. 10-26

allows Equations 10-25 and 10-26 to be combined so as to isolate a_2 with p_2 in an equation involving nothing more than the string-coupled pendulum's dimensions and the gravitational constant:

$$a_2 = -\frac{G}{(L + S \cdot Q / H)} \cdot p_2$$ Eqa. 10-27

With Equations 10-7 and 10-27 in hand we can appeal to the analysis that closed our previous article on X-motion, Equations 9-34 to 9-39, to say that the frequencies for the simple harmonic motion specified in these two equations have been identified as the internal frequencies of the bi-harmonic oscillator equations that were set out in Articles 2 and 3:

$$\omega_{1Y} = \sqrt{\frac{G}{L + Q \cdot S / H}}$$ Eqa. 3-8

$$\omega_{2Y} = \sqrt{\frac{G}{L + Q}}$$ Eqa. 3-3

This means that our geometric analyses in Article 3 regarding Y-motion have been validated by application of our principles of analysis.

Article 11: Recapitulation. Articles 9 and 10 have now confirmed earlier speculation as to the natural frequencies ω_1 and ω_2 underlying both X- and Y-motions of string-coupled pendula. These are tabulated below.

	Y-motion	X-motion
ω_1	$\omega_{1Y} = \sqrt{\frac{G}{L + Q \cdot S/H}}$	$\omega_{1X} = \sqrt{\frac{G}{L}}$
ω_2	$\omega_{2Y} = \sqrt{\frac{G}{L + Q}}$	$\omega_{2X} = \sqrt{\frac{G}{L + Q}}$

Tbl. 11-1

Here we see that the same lesser frequency $\omega_2 = \omega_0$ underlies both X- and Y-motion,

$$\omega_0 = \sqrt{\frac{G}{L + Q}} \qquad \text{Eqa. 11-1}$$

while the greater frequency ω_1 is unique for motions in either coordinate direction:

$$\omega_X = \sqrt{\frac{G}{L}} \qquad \text{Eqa. 11-2}$$

$$\omega_Y = \sqrt{\frac{G}{L + Q \cdot S/H}} \qquad \text{Eqa. 11-3}$$

Restricting ourselves to those motions arising solely from initial displacements of the pendulum weights, D_{1X}, D_{2X}, D_{1Y}, and D_{2Y}, we can elaborate Equations 2-3 and 2-4 as follows:

Eqa. 11-4

$$\begin{aligned} P_1(t) = {} & \hat{i} \cdot D_{1X} \cdot \cos\left(\frac{\omega_X + \omega_0}{2} \cdot t\right) \cdot \cos\left(\frac{\omega_X - \omega_0}{2} \cdot t\right) \\ & + \hat{j} \cdot D_{1Y} \cdot \cos\left(\frac{\omega_Y + \omega_0}{2} \cdot t\right) \cdot \cos\left(\frac{\omega_Y - \omega_0}{2} \cdot t\right) \\ & + \hat{i} \cdot D_{2X} \cdot \sin\left(\frac{\omega_X + \omega_0}{2} \cdot t\right) \cdot \sin\left(\frac{\omega_X - \omega_0}{2} \cdot t\right) \\ & + \hat{j} \cdot D_{2Y} \cdot \sin\left(\frac{\omega_Y + \omega_0}{2} \cdot t\right) \cdot \sin\left(\frac{\omega_Y - \omega_0}{2} \cdot t\right) \end{aligned}$$

Eqa. 11-5

$$\begin{aligned} P_2(t) = {} & \hat{i} \cdot D_{2X} \cdot \cos\left(\frac{\omega_X + \omega_0}{2} \cdot t\right) \cdot \cos\left(\frac{\omega_X - \omega_0}{2} \cdot t\right) \\ & + \hat{j} \cdot D_{2Y} \cdot \cos\left(\frac{\omega_Y + \omega_0}{2} \cdot t\right) \cdot \cos\left(\frac{\omega_Y - \omega_0}{2} \cdot t\right) \\ & + \hat{i} \cdot D_{1X} \cdot \sin\left(\frac{\omega_X + \omega_0}{2} \cdot t\right) \cdot \sin\left(\frac{\omega_X - \omega_0}{2} \cdot t\right) \\ & + \hat{j} \cdot D_{1Y} \cdot \sin\left(\frac{\omega_Y + \omega_0}{2} \cdot t\right) \cdot \sin\left(\frac{\omega_Y - \omega_0}{2} \cdot t\right) \end{aligned}$$

where $\hat{\imath}$ and $\hat{\jmath}$ are vectors of unit length along the X-and Y-axes.

Appendix: The differential system. Equations 11-4 and 11-5 are offered as a closed-form solution to the string-coupled pendulum problem: for any initial displacements, the positions of the pendulum weights can be determined at any point in time t. Now that the natural frequencies of the string-coupled pendulum are known, it remains to identify these frequencies in terms a differential system allowing for continuous emulation of the pendulum's behaviors.

Where a complicated system's closed-form solution is unattainable (as is often the case) its differential representation can provide the only means of establishing its behaviors. These representations are also informative insofar as they require an understanding of the system's internal mechanics. For example: Figure 7-1's depiction of the relationship between the pendulum weight's position P and its acceleration A is used to complete the signal path at Figure A-1's right.

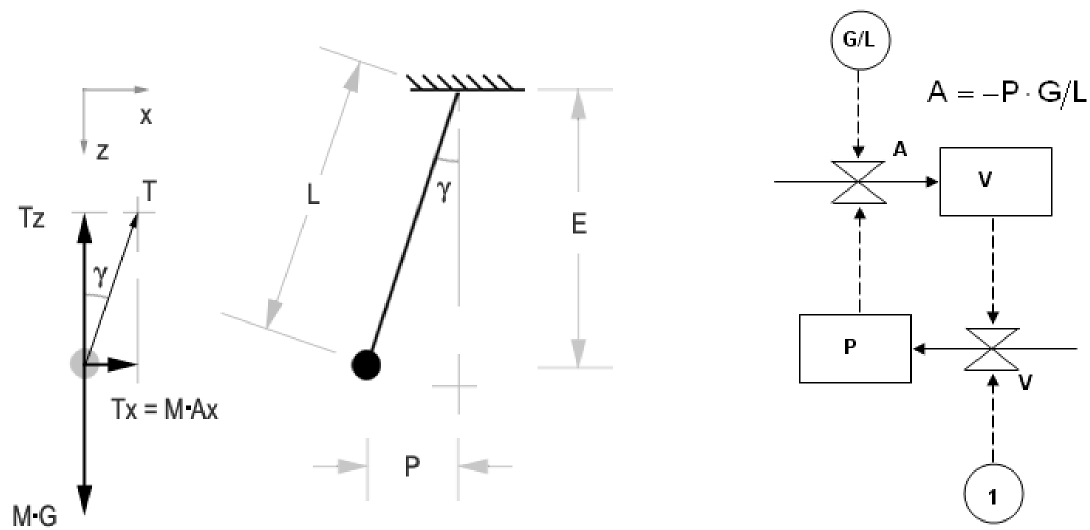


Fig. A-1

Figure A-1 particularizes pendula in terms of generic simple harmonic oscillators, all of which exhibit this same signal path. Here we see velocity V as both the integration of acceleration A, and position P's rate of change. Thus a gain of unity links the rate V with the state variable V. The gain –G/L linking P with A assures that this simple representation of the relationships among A, V, and P will oscillate with a frequency of $\sqrt{G/L}$.

Representing our string-coupled pendulum in this manner reveals that the accelerations A in either coordinate direction are linked to the positions P of both pendulum weights:

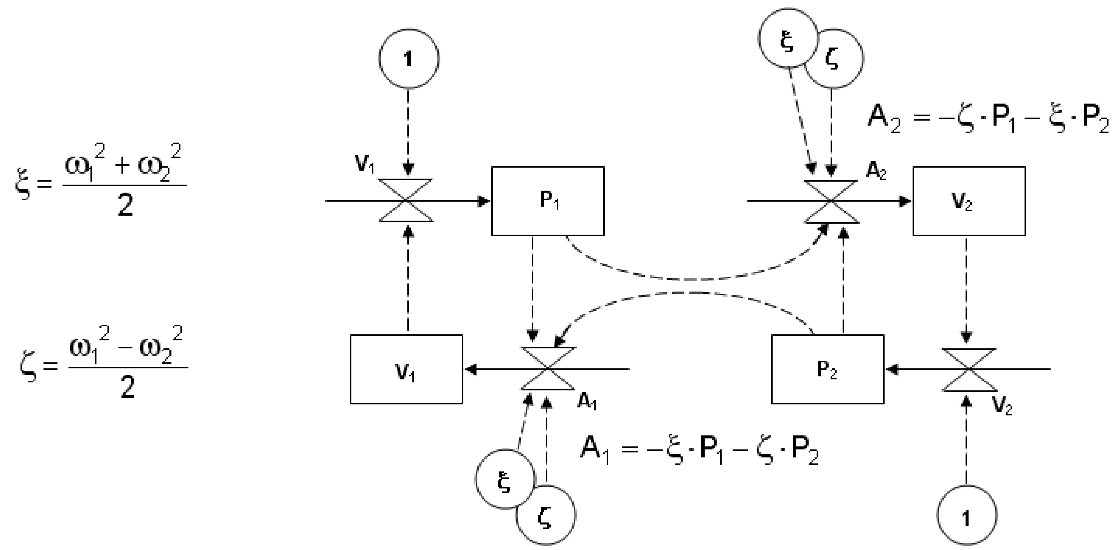


Fig. A-2

This classical differential form of the equation for a bi-harmonic oscillator is governed a system such as …

$$A_1 = -\xi \cdot P_1 - \zeta \cdot P_2 \qquad \text{Eqa. A-1}$$

$$A_2 = -\zeta \cdot P_1 - \xi \cdot P_2 \qquad \text{Eqa. A-2}$$

which would conceals the natural frequencies of whatever system they describe thusly:

$$\xi = \frac{\omega_1^{\,2} + \omega_2^{\,2}}{2} \qquad \text{Eqa. A-3}$$

$$\zeta = \frac{\omega_1^{\,2} - \omega_2^{\,2}}{2} \qquad \text{Eqa. A-4}$$

Formal development of Equations A-1 through A-4 will complete our exposition of the pendulum problem.

Equations 2-1 and 2-2 generate bi-harmonic oscillations in either the X- or Y-directions when the first pendulum weight is given an initial displacement of D and the second pendulum begins from rest at time t=0. A certain change of variables will make these equations easier to handle in the development to follow:

$$\alpha = \frac{\omega_1 + \omega_2}{2} \qquad \text{Eqa. A-5}$$

$$\beta = \frac{\omega_1 - \omega_2}{2} \qquad \text{Eqa. A-6}$$

$$\rho_1 = \frac{P_1}{D} = \cos(\alpha \cdot t) \cdot \cos(\beta \cdot t) \qquad \text{Eqa. A-7}$$

$$\rho_2 = \frac{P_2}{D} = \sin(\alpha \cdot t) \cdot \sin(\beta \cdot t) \qquad \text{Eqa. A-8}$$

We also anticipate using the derivatives for sine and cosine functions …

$$\frac{\partial \sin(t)}{\partial t} = \cos(t) \qquad \frac{\partial \cos(t)}{\partial t} = -\sin(t)$$

as well as the chain rule: $\frac{\partial uv}{\partial t} = u\frac{\partial v}{\partial t} + v\frac{\partial u}{\partial t}$

These assertions allow further simplifying substitutions for Equation A-7 …

$$u = \cos(\alpha \cdot t) \qquad \text{Eqa. A-9}$$

$$v = \cos(\beta \cdot t) \qquad \text{Eqa. A-10}$$

$$\rho_1 = u \cdot v \qquad \text{Eqa. A-11}$$

$$\frac{\partial \rho_1}{\partial t} = u\frac{\partial v}{\partial t} + v\frac{\partial u}{\partial t} \qquad \text{Eqa. A-12}$$

When the trigonometric identities of u, v, and their derivatives are substituted into Equation A-12 we arrive at ρ_1's velocity:

$$\frac{\partial \rho_1}{\partial t} = -\beta \sin(\beta \cdot t)\cos(\alpha \cdot t) - \alpha \sin(\alpha \cdot t)\cos(\beta \cdot t) \qquad \text{Eqa. A-13}$$

And these same identities then allow computation of ρ_1's acceleration:

$$\begin{aligned}\frac{\partial^2 \rho_1}{\partial t^2} &= [-\beta \sin(\beta \cdot t)] \cdot [-\alpha \sin(\alpha \cdot t)] \\ &+ [\cos(\alpha \cdot t)] \cdot [-\beta^2 \cos(\beta \cdot t)] \\ &+ [-\alpha \sin(\alpha \cdot t)] \cdot [-\beta \sin(\beta \cdot t)] \\ &+ [\cos(\beta \cdot t)] \cdot [-\alpha^2 \cos(\alpha \cdot t)]\end{aligned} \qquad \text{Eqa. A-14}$$

Here we see elements of Equations A-7 and A-8 for ρ_1 and ρ_2 emerging on the right side of Equation A-14. Combining these three equations reveals that ρ_1's second derivative A_1/D involves a simple function of ρ_1 and ρ_2 that bring us close to the form of Equation A-1:

$$\frac{\partial^2 \rho_1}{\partial t^2} = \left(-\alpha^2 - \beta^2\right) \cdot \rho_1 - 2 \cdot \alpha \cdot \beta \cdot \rho_2 \qquad \text{Eqa. A-15}$$

Reference to Equations A-5 and A-6 allow us to evaluate the α and β terms in Equation A-15:

$$\begin{aligned}-\alpha^2 - \beta^2 &= -\left[\frac{\omega_1^2 + 2 \cdot \omega_1 \cdot \omega_2 + \omega_2^2}{4} + \frac{\omega_1^2 - 2 \cdot \omega_1 \cdot \omega_2 + \omega_2^2}{4}\right] \\ &= -\frac{\omega_1^2 + \omega_2^2}{2}\end{aligned} \qquad \text{Eqa. A-16}$$

$$\begin{aligned}-2 \cdot \alpha \cdot \beta &= -2 \cdot \left[\frac{\omega_1^2 - \omega_2^2}{4}\right] \\ &= -\frac{\omega_1^2 - \omega_2^2}{2}\end{aligned} \qquad \text{Eqa. A-17}$$

Substituting the identities for $-\alpha^2-\beta^2$ and $-2\alpha\beta$ established in Equations A-16 and A-17 into Equation A-15 …

$$\frac{\partial^2\rho_1}{\partial t^2} = -\frac{{\omega_1}^2 + {\omega_2}^2}{2}\cdot\rho_1 - \frac{{\omega_1}^2 - {\omega_2}^2}{2}\cdot\rho_2 \qquad \text{Eqa. A-18}$$

validates our expectation that Equations A-1, A-3, and A-4 operate together in describing the dynamics of string-coupled pendula. A similar development proceeding from Equation A-8 will validate Equation A-2's inclusion in a complete specification of these dynamics.

Parallel development of ρ_2 begins at Equation A-9 with a re-specification of the variables used in applying the chain rule …

$$u = \sin(\alpha\cdot t) \qquad \text{Eqa. A-19}$$

$$v = \sin(\beta\cdot t) \qquad \text{Eqa. A-20}$$

$$\rho_2 = u\cdot v \qquad \text{Eqa. A-21}$$

$$\frac{\partial\rho_2}{\partial t} = u\frac{\partial v}{\partial t} + v\frac{\partial u}{\partial t} \qquad \text{Eqa. A-22}$$

that quickly resolves to ρ_2's velocity:

$$\frac{\partial\rho_2}{\partial t} = \sin(\alpha\cdot t)\cdot[\beta\cdot\cos(\beta\cdot t)] + \sin(\beta\cdot t)\cdot[\alpha\cdot\cos(\alpha\cdot t)] \qquad \text{Eqa. A-23}$$

Once again, these same identities allow computation of ρ_2's acceleration …

$$\begin{aligned}\frac{\partial^2\rho_2}{\partial t^2} &= \sin(\alpha\cdot t)\cdot\left[-\beta^2\sin(\beta\cdot t)\right] \\ &+ [\beta\cdot\cos(\beta\cdot t)]\cdot[\alpha\cdot\cos(\alpha\cdot t)] \\ &+ \sin(\beta\cdot t)\cdot\left[-\alpha^2\sin(\alpha\cdot t)\right] \\ &+ \alpha\cdot\cos(\alpha\cdot t)\cdot[\beta\cdot\cos(\beta\cdot t)]\end{aligned} \qquad \text{Eqa. A-24}$$

where we see elements of Equations A-7 and A-8 for ρ_1 and ρ_2 emerging on the right side of Equation A-24. Combining these three equations reveals that ρ_2's second derivative A_2/D also involves simple a function of ρ_1 and ρ_2 that bring us closer to the form of Equation A-2:

$$\frac{\partial^2 \rho_2}{\partial t^2} = -2 \cdot \alpha \cdot \beta \cdot \cos(\alpha \cdot t) \cdot \cos(\beta \cdot t) - \left(\alpha^2 + \beta^2\right) \cdot \sin(\alpha \cdot t) \cdot \sin(\beta \cdot t) \qquad \text{Eqa. A-25}$$

Returning to Equations A-5 and A-6, we can evaluate the α and β terms in Equation A-25 by substituting the identities for $-\alpha^2 - \beta^2$ and $-2\alpha\beta$ established in Equations A-16 and A-17 …

$$\frac{\partial^2 \rho_2}{\partial t^2} = -\frac{\omega_1{}^2 - \omega_2{}^2}{2} \cdot \rho_1 - \frac{\omega_1{}^2 + \omega_2{}^2}{2} \cdot \rho_2 \qquad \text{Eqa. A-26}$$

thus validating the simultaneous operations of Equations A-2, A-3, and A-4.

Moving on to particularize these expressions of ξ and ζ for Y-motion, we substitute the right-hand sides of Equations 11-1 and 11-3 into Equations A-3 and A-4:

$$\xi_Y = \frac{\dfrac{G}{L + Q \cdot S/H} + \dfrac{G}{L + Q}}{2} \qquad \text{Eqa. A-27}$$

$$\zeta_Y = \frac{\dfrac{G}{L + Q \cdot S/H} - \dfrac{G}{L + Q}}{2} \qquad \text{Eqa. A-28}$$

Setting the respective numerators of these equations on a common denominator brings us closer to their geometric identities:

$$\xi_Y = \frac{G}{2} \cdot \frac{H \cdot (L + Q) + H \cdot (L + Q \cdot S/H)}{H \cdot (L + Q \cdot S/H) \cdot (L + Q)} \qquad \text{Eqa. A-29}$$

$$\zeta_Y = \frac{G}{2} \cdot \frac{H \cdot (L + Q) - H \cdot (L + Q \cdot S/H)}{H \cdot (L + Q \cdot S/H) \cdot (L + Q)} \qquad \text{Eqa. A-30}$$

Proceeding to develop Equation A-29 …

$$\xi_Y = \frac{G}{2} \cdot \frac{2 \cdot L + Q \cdot (H + S)}{L^2 H \cdot + L \cdot Q \cdot S + L \cdot Q \cdot H + Q^2 S}$$

Eqa. A-31

$$\xi_Y = G \cdot \frac{L \cdot H + Q \cdot \left(\frac{H+S}{2}\right)}{H \cdot L^2 + L \cdot Q \cdot S/2 + L \cdot Q \cdot S/2 + H \cdot L \cdot Q/2 + H \cdot L \cdot Q/2 + Q^2 S}$$

Eqa. A-32

$$\xi_Y = \frac{G \cdot L \cdot H + Q \cdot \frac{H+S}{2}}{L \cdot \left(H \cdot L + Q \cdot \frac{(H+S)}{2}\right) + Q \cdot \left(L \cdot \frac{(H+S)}{2} + Q \cdot S\right)}$$

Eqa. A-33

we arrive at expressions for areas in the X-Z plane that were marked-out in Figure 5-1:

$$\xi_Y = \frac{G \cdot \vartheta}{L \cdot \vartheta + Q \cdot \lambda}$$

Eqa. A-34

A parallel development for Equation A-30 anticipates the same treatment of the denominator as was carried-out in Equation A-32 …

$$\zeta_Y = \frac{G}{2} \cdot \frac{Q \cdot (H - S)}{L^2 H \cdot + L \cdot Q \cdot S + L \cdot Q \cdot H + Q^2 S}$$

Eqa. A-35

leading to …

$$\zeta_Y = \frac{Q \cdot \frac{H-S}{2}}{L \cdot \left(H \cdot L + Q \cdot \frac{(H+S)}{2}\right) + Q \cdot \left(L \cdot \frac{(H+S)}{2} + Q \cdot S\right)}$$

Eqa. A-36

which also relates to expressions for areas in the X-Z plane that were marked-out in Figure 5-1:

$$\zeta_Y = \frac{G \cdot \varphi}{L \cdot \vartheta + Q \cdot \lambda} \qquad \text{Eqa. A-37}$$

Here we might note that the ratio of ξ_Y to ζ_Y equals Equation 5-18's approximation for the number of pulses per half cycle exhibited in Y-motion, ϑ/φ.

Calculating ξ and ζ for X-motion begins by substituting the expressions for ω_X and ω_0 of Equations 11-1 and 11-2 into Equations A-3 and A-4 …

$$\xi_X = \frac{\frac{G}{L} + \frac{G}{L+Q}}{2} \qquad \text{Eqa. A-38}$$

$$\zeta_X = \frac{\frac{G}{L} - \frac{G}{L+Q}}{2} \qquad \text{Eqa. A-39}$$

which leads to immediate simplifications having no need for further reductions:

$$\xi_X = \frac{G}{2} \cdot \frac{2 \cdot L + Q}{L(L+Q)} \qquad \text{Eqa. A-40}$$

$$\zeta_X = \frac{G}{2} \cdot \frac{Q}{L(L+Q)} \qquad \text{Eqa. A-41}$$

It remains only to note that the ratio of ξ_X to ζ_X tends to approximate Equation 6-8's determination of the number of pulses per half cycle exhibited in X-motion:

$$\frac{\xi_X}{\zeta_X} = 1 + 2 \cdot \frac{L}{Q} \overset{?}{=} \frac{1}{2} + \frac{L}{Q} + \sqrt{\frac{L^2}{Q^2} + \frac{L}{Q}} \qquad \text{Eqa. A-42}$$

To the extent that the equality on the right of Equation A-42 might be true, the following equation must be valid:

$$\frac{1}{2}+\frac{L}{Q}\approx\sqrt{\frac{L^2}{Q^2}+\frac{L}{Q}} \qquad \text{Eqa. A-43}$$

Squaring both sides, we see that the equality holds insofar as the ratio 1:4 is insignificant in comparison to the ratio L:Q.

$$\frac{1}{4}+\frac{L}{Q}+\frac{L^2}{Q^2}\approx\frac{L^2}{Q^2}+\frac{L}{Q} \qquad \text{Eqa. A-44}$$