Multi-dimensional Hyperbolae
and
Loci Technical Indifference

Abstract

Demonstrations of neoclassical causality seldom originate in utility tradeoffs defined to the extent of specifying a particular functional form. The function must present certain properties, especially diminishing marginal utility, which are then demonstrated sufficient for the deductions to be made respecting the behavior of economic systems. SFEcon's specific choice of hyperbolic utility tradeoffs reflects our finding that this functional form is as uniquely suited to macroeconomic causality as, say, the Schrödinger Equation is to elementary chemistry, or as the golden mean is to biology. Empirical and philosophical cases for this conviction are developed at sfecon.com. Our purpose here is to simply to draw out all the analytic capabilities that the hyperbola might provide to economic modeling.

Marginalist criteria for productive optimality are (rightly or wrongly) thought to determine an economic being's optimal operating decisions, viz.: his budgetary expenditure, distribution of the budget among productive factors, optimal output, and maximal profits. It is speculated that these particulars are entirely determined by 1) a presumption that marginal revenues equal marginal costs; 2) knowledge of the geometry of production tradeoffs; and 3) an awareness of the price environment in which operating decisions are made. While these matters have been proven to the satisfaction of most, their elucidation has yet to reveal a general and precise quantitative counterpart for all that their geometric presentation entails. This paper examines hyperbolic expressions of productive indifference in terms of their ability to answer, in closed-form and for any number of inputs, each of the questions comprised in the marginalist oeuvre.

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and
Loci of Technical Indifference

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1: Hyperbolic Descriptions of Utility Tradeoffs

Figure 1-1 shows how a two-dimensional, negative hyperbolic form might be used to express the diminishing marginal utility of a single input E in the production of an output Y. Parameters Z and U locate the production function's origin relative to what might be called the natural origin of the hyperbolic form, i.e.: the intersection point of the all hyperbola's asymptotes. These parameters can be shown to fully describe the relationship between Y and E.

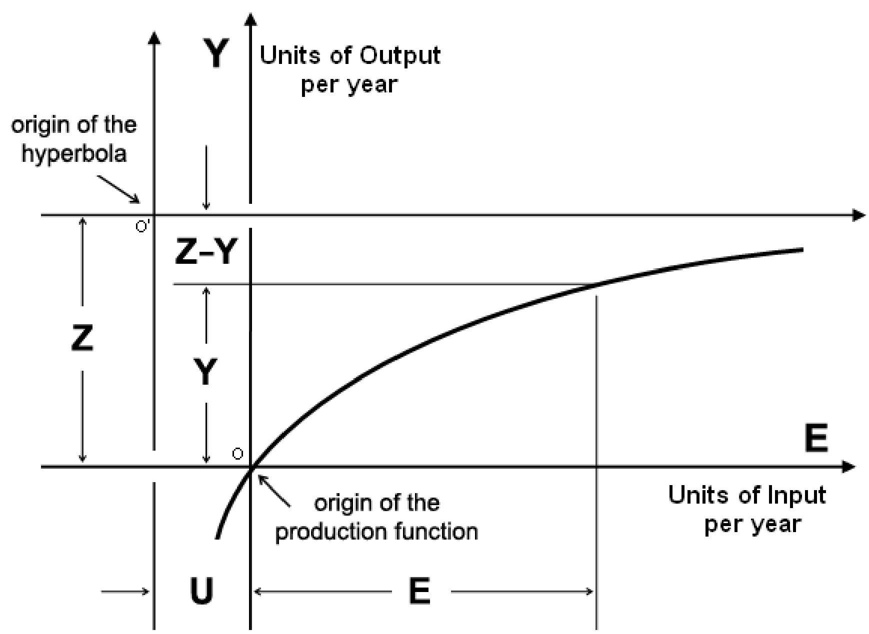

Fig. 1-1

Since output must vanish when input vanishes, the production function's origin point [Z,U] must lie on a general hyperbolic form governed by some parameter a:

$$Z = \frac{-a}{U} \quad \text{Eqa. 1-1}$$

And the requirement that any other combination of Y and E must lie on the hyperbola defined by the same parameter **a** yields a similar equation:

$$Z - Y = \frac{-a}{U + E} \quad \text{Eqa. 1-2}$$

Equations 1-1 and 1-2 can now be solved to eliminate a,

$$\frac{Z-Y}{Z}=\frac{U}{U+E} \qquad \text{Eqa. 1-3}$$

and rearranged to yield the desired relationship between output Y and input E:

$$Y=Z\cdot\left[1-\frac{U}{U+E}\right] \qquad \text{Eqa. 1-4}$$

Figure 1-2 portrays this same relationship in the three dimensions needed for the case of two inputs creating the product.

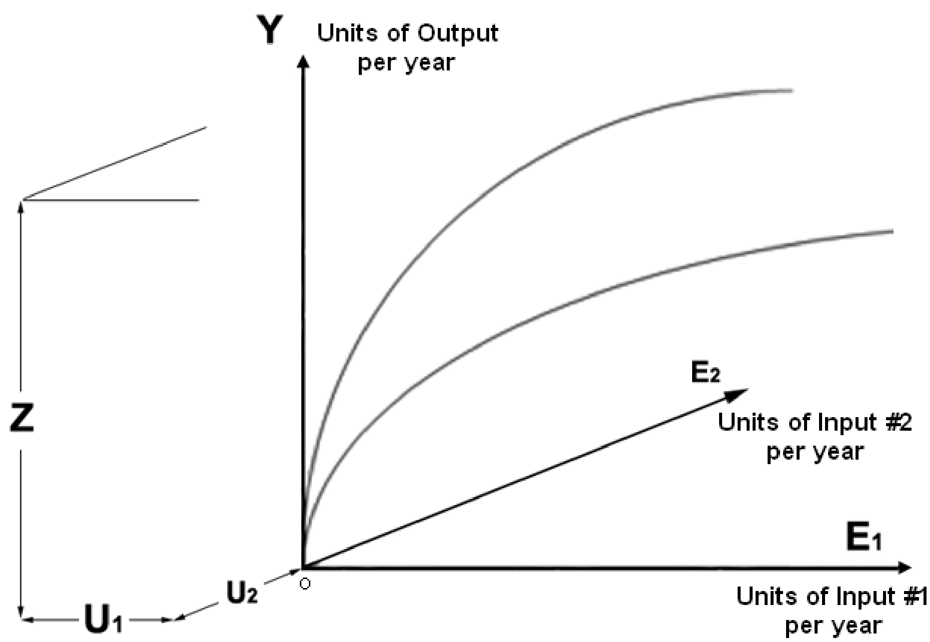

Fig. 1-2

As this figure exhausts our ability to visualize productive indifference, we must now engage a purely mathematical model to describe the general case of N factors creating the output. Equations 1-5 through 1-8 recapitulate the analysis in Equations 1-1 through 1-4 for this general case.

$$Z=\frac{-a}{U_1\cdot U_2\cdot\ldots\cdot U_N} \qquad \text{Eqa. 1-5}$$

$$Z - Y = \frac{-a}{(U_1 + E_1)\cdot(U_2 + E_2)\cdot\ldots\cdot(U_N + E_N)}$$ Eqa. 1-6

$$\frac{Z - Y}{Z} = \frac{U_1 \cdot U_2 \cdot\ldots\cdot U_N}{(U_1 + E_1)\cdot(U_2 + E_2)\cdot\ldots\cdot(U_N + E_N)}$$ Eqa. 1-7

$$Y = Z\cdot\left[1 - \frac{U_1 \cdot U_2 \cdot\ldots\cdot U_N}{(U_1 + E_1)\cdot(U_2 + E_2)\cdot\ldots\cdot(U_N + E_N)}\right]$$ Eqa. 1-8

Equation 1-8 parameterizes of the relation between an output Y and inputs E_J in terms of a utility set [Z,U_J]. This set describes the vector offset between the production function's origin and the point of intersection of all asymptotes to the hyperbolic form constituting the locus of technical optima.

2: The Polynomial Factoring Problem

Marginalist discourse describes economic optimality in terms of a relation between prices and the gradient of a utility function at its operating point [Y,E_J]. Differentiating the production function of Equation 1-8 with respect to an input J yields the following expression of marginal product for hyperbolic expressions of indifference:

$$\frac{\partial Y}{\partial E_J} = \frac{Z - Y}{U_J + E_J}$$ Eqa. 2-1

Per the fundamental theorem of calculus, optimal use of an input commodity J occurs when J's marginal product equals the ratio of marginal costs (i.e.: the input's price P_J) to marginal revenues (i.e.: the output's price π). Denoting the optimal instance of the [Y,E_J] set as [ξ,Q_J], we equate the price ratio with the marginal products of Equation 2-1:

$$\frac{Z-\xi}{U_J+Q_J}=\frac{P_J}{\pi} \qquad \text{Eqa. 2-2}$$

Elaborating Equation 2-2 across all of a product's inputs J=1…N presents a central equality ζ uniting the price, utility parameter, and physical flow associated with each axis of the utility function by which an economic actor is defined.

$$\begin{aligned}
\zeta &= \pi\cdot(Z-\xi) \\
&= P_1\cdot(U_1+Q_1) \\
&= P_2\cdot(U_2+Q_2) \\
&\vdots \\
&= P_N\cdot(U_N+Q_N)
\end{aligned} \qquad \text{Eqa. 2-3}$$

For want of a more descriptive name, the parameter ζ is referred-to as a 'financial discriminant'. It constitutes the primary financial descriptor of an economic sector I in an economy K. If the optimal ζ_{IK} is known then sector IK can, by way of Equation 2-3, determine its optimal asset usage rates Q_{IJK} from the shape of its production tradeoffs U_{IJK} and prices P_{JK}.

Determination of ζ requires that all elements of Equations 2-3 cooperate in an exact solution to the production function of Equation 1-8. Introducing a unit ratio P_J/P_J to each term of Equation 1-7 …

$$\frac{\pi\cdot(Z-\xi)}{\pi\cdot Z}=\frac{P_1\cdot U_1}{P_1\cdot(U_1+Q_1)}\cdot\frac{P_2\cdot U_2}{P_2\cdot(U_2+Q_2)}\cdot\ldots\cdot\frac{P_N\cdot U_N}{P_N\cdot(U_N+Q_N)} \qquad \text{Eqa. 2-4}$$

allows the substitution of ζ into each term of the above:

$$\frac{\zeta}{\pi\cdot Z}=\frac{P_1\cdot U_1}{\zeta}\cdot\frac{P_2\cdot U_2}{\zeta}\cdot\ldots\cdot\frac{P_N\cdot U_N}{\zeta} \qquad \text{Eqa. 2-5}$$

Cross-multiplying over the equality isolates ζ:

$$\zeta^{N+1} = (\pi \cdot Z \cdot P_1 \cdot U_1 \cdot P_2 \cdot U_2 \cdot \ldots \cdot P_N \cdot U_N) \qquad \text{Eqa. 2-6}$$

Solving for ζ yields:

$$\zeta = (\pi \cdot Z \cdot P_1 \cdot U_1 \cdot P_2 \cdot U_2 \cdot \ldots \cdot P_N \cdot U_N)^{1/(N+1)} \qquad \text{Eqa. 2-7}$$

where we see economics' classic polynomial factoring problem reduced to the extraction of a higher-ordered root. Equations 2-3 and 2-7 demonstrate that [Z,U_J] and [π,P_J] determine [ξ,Q_J] in mathematically closed-form.

3: Budget Constraints and the Expansion Path

To this point we have epitomized the interaction of a price vector with production tradeoffs in the financial discriminant ζ. Now it would be useful to solidify our examination of the hyperbolic form by locating ζ amid the more accepted analytic pattern for discerning economic optima. This pattern usually begins with an exterior specification of a budget constraint, σ:

$$\sigma = \sum_{J=1}^{N} \left(P_J \cdot Q_J\right) \qquad \text{Eqa. 3-1}$$

The analysis then proceeds to determine the spectrum of inputs Q_J that produce the greatest output Y under the budget constraint σ – a process generally envisioned as in Figure 3-1.

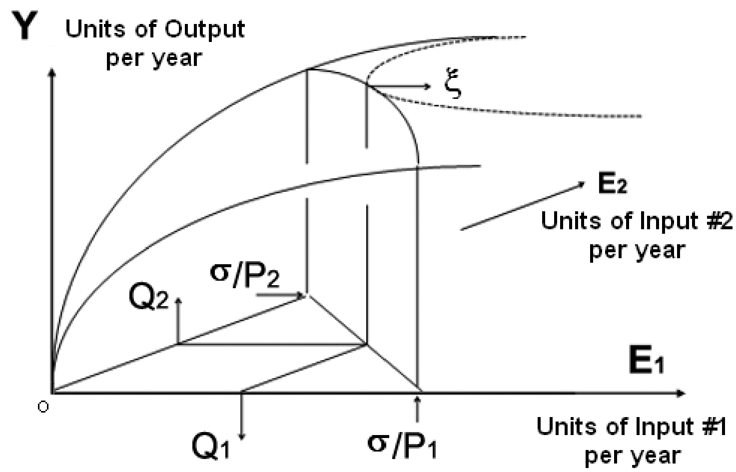

Fig. 3-1

As currently accomplished for, say, the Cobb-Douglas Production Function, disclosure of the optimal distribution of σ among inputs J involves a Lagrangian transformation requiring solution to a linear system of N equations in N unknowns. For hyperbolic expressions of productive indifference, this task can be accomplished in closed-form. All such analyses begin with the same geometric criteria: maximal Y occurs for the spectrum of inputs $E_1, E_2, \ldots, E_N$ where marginal rates of technical substitution equal the slope of the budget constraint:

$$-\frac{\partial E_I}{\partial E_J} = -\frac{P_J}{P_I} \qquad \text{Eqa. 3-2}$$

Since marginal rates of technical substitution derive from marginal products,

$$-\frac{\partial E_I}{\partial E_J} = -\frac{\partial Y / \partial E_J}{\partial Y / \partial E_I} \qquad \text{Eqa. 3-3}$$

they can be readily inferred from Equation 2-1:

$$-\frac{\partial E_I}{\partial E_J} = -\frac{(Z - Y)/(U_J - E_J)}{(Z - Y)/(U_I - E_I)} \qquad \text{Eqa. 3-4}$$

Combining Equations 3-2 and 3-4 expresses our geometric criteria in terms of hyperbolic systems:

$$P_I \cdot (U_I + E_I) \;=\; P_J \cdot (U_J + E_J) \qquad \text{Eqa. 3-5}$$

This linear relationship among rates of input E_J under a budget constraint is appropriate to the commonplace notion of expansion paths.

Equation 3-5 is a re-expression of Equation 2-3 where the optimal operating decision $Q_1, Q_2, \ldots, Q_N$. has been replaced by the general notation for an operating point $E_1, E_2, \ldots, E_N$. When the E_J's equal the optimal Q_J's the financial discriminant ζ becomes the identity of all the quantities equated in Equation 3-5:

$$\zeta = P_J \cdot (U_J + Q_J) \qquad \text{Eqa. 3-6}$$

This observation permits us to isolate the budget constraint σ by adding all N Equations 3-6:

$$N \cdot \zeta = \sum_{J=1}^{N} (P_J \cdot U_J) + \sum_{J=1}^{N} (P_J \cdot Q_J) \qquad \text{Eqa. 3-7}$$

Reference to Equation 3-1 shows that the budget σ has emerged in Equation 3-7's right-most term, while the interaction of prices with the shape of production tradeoffs is entirely contained in the middle term:

$$\overline{PU} = \sum_{J=1}^{N} (P_J \cdot U_J) \qquad \text{Eqa. 3-8}$$

Equations 3-7 and 3-8 entirely define ζ:

$$\zeta = \frac{\sigma + \overline{PU}}{N} \qquad \text{Eqa. 3-9}$$

Equation 3-6 can be re-arranged to determine the optimal operating decision $Q_1, Q_2, \ldots, Q_N$ in terms of ζ:

$$Q_J = \frac{\zeta}{P_J} - U_J \qquad \text{Eqa. 3-10}$$

Eliminating ζ from Equations 3-9 and 3-10 results in a closed-form determination of Q_J in terms of the budget constraint σ, input prices P_J and the shapes of production tradeoffs U_J:

$$Q_J = \frac{\sigma + \overline{PU}}{N \cdot P_J} - U_J \qquad \text{Eqa. 3-11}$$

4: The Total Cost Curve:

Our examination of operating decisions given exterior specification of a budget constraint σ yielded a spectrum of input rates Q_1, Q_2, … , Q_N that is optimal in the sense of producing the greatest possible rate of output ξ. Our task here is to derive and optimize the functional relationship between σ and ξ in order to create the total cost curve of Figure 4-1.

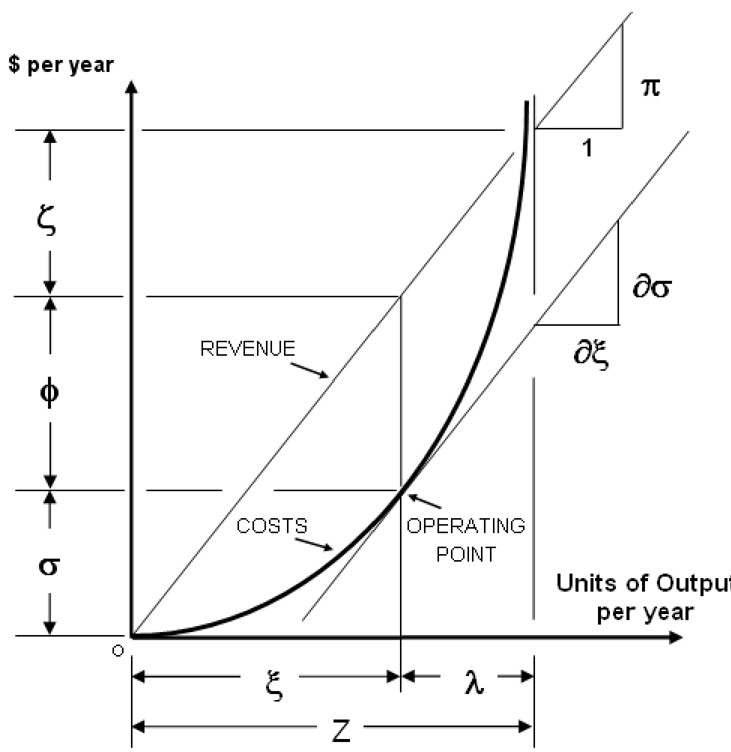

Fig. 4-1

We begin by eliminating the output ξ's price π from Equation 2-4's presentation of the production function:

$$\frac{(Z-\xi)}{Z} = \frac{P_1 \cdot U_1}{P_1 \cdot (U_1 + Q_1)} \cdot \frac{P_2 \cdot U_2}{P_2 \cdot (U_2 + Q_2)} \cdot \ldots \cdot \frac{P_N \cdot U_N}{P_N \cdot (U_N + Q_N)} \qquad \text{Eqa. 4-1}$$

Equation 4-1 can be simplified by reference to Equation 2-3,

$$\begin{aligned} \zeta &= \pi \cdot (Z-\xi) \\ &= P_1 \cdot (U_1 + Q_1) \\ &= P_2 \cdot (U_2 + Q_2) \\ &\vdots \\ &= P_N \cdot (U_N + Q_N) \end{aligned} \qquad \text{Eqa. 2-3}$$

which shows that a computation of the financial discriminant ζ appears in each denominator on this equation's right-hand side:

$$\frac{(Z-\xi)}{Z} = \frac{P_1 \cdot U_1}{\zeta} \cdot \frac{P_2 \cdot U_2}{\zeta} \cdot \ldots \cdot \frac{P_N \cdot U_N}{\zeta} \qquad \text{Eqa. 4-2}$$

When the interaction between factor prices and production tradeoffs is consolidated in Equation 4-3,

$$\overline{\overline{PU}} = \prod_{J=1}^{N} \left(P_J \cdot U_J\right) \qquad \text{Eqa. 4-3}$$

Equation 4-2 can be re-stated in a compact form …

$$\frac{(Z-\xi)}{Z} = \frac{\overline{\overline{PU}}}{\zeta^N} \qquad \text{Eqa. 4-4}$$

that allows isolation of ζ:

$$\zeta^{N} = \frac{Z \cdot \overline{\overline{PU}}}{(Z - \xi)} \qquad \text{Eqa. 4-5}$$

Another reference to ζ can be recalled from our earlier analysis of the budget constraint:

$$\zeta = \frac{\sigma + \overline{PU}}{N} \qquad \text{Eqa. 3-9}$$

This equation gives us a relationship between ζ the budget σ in terms of another epitome of the interaction between factor prices and production tradeoffs, viz.:

$$\overline{PU} = \sum_{J=1}^{N} \left(P_J \cdot U_J\right) \qquad \text{Eqa. 3-8}$$

We can now eliminate ζ 's from Equations 3-9 and 4-5 to establish the desired relationship between a budget σ and the greatest output ξ that it can support:

$$\sigma = N \cdot \left[\frac{Z \cdot \overline{\overline{PU}}}{Z - \xi}\right]^{\frac{1}{N}} - \overline{PU} \qquad \text{Eqa. 4-6}$$

Figure 4-1 superimposes a revenue curve on Equation 4-6’s cost curve by simply erecting a straight line through the origin at a slope equal the output's price π. This presentation allows us to visualize the final aspect of optimality in which a combination of σ and ξ is chosen so as to maximize operating profits ϕ. Once again, the fundamental theorem of calculus tells that maximal profits occur where marginal revenue equals marginal costs:

$$\pi = \frac{\partial \sigma}{\partial \xi} \qquad \text{Eqa. 4-7}$$

The cost curve, Equation 4-6, is easily differentiated,

$$\frac{\partial\sigma}{\partial\xi} = \frac{1}{Z-\xi}\cdot\left[\frac{Z\cdot\overline{\overline{PU}}}{Z-\xi}\right]^{\frac{1}{N}} \qquad \text{Eqa. 4-8}$$

and the differential expression is eliminated by combining Equations 4-7 and 4-8:

$$\pi = \frac{1}{Z-\xi}\cdot\left[\frac{Z\cdot\overline{\overline{PU}}}{Z-\xi}\right]^{\frac{1}{N}} \qquad \text{Eqa. 4-9}$$

Rearranging Equation 4-9 presents us with an equation that can be reduced to an identity confirming our earlier assertions relating to solution of the polynomial factoring problem:

$$\pi\cdot(Z-\xi) = \left[\frac{\pi\cdot Z\cdot\overline{\overline{PU}}}{\pi\cdot(Z-\xi)}\right]^{\frac{1}{N}} \qquad \text{Eqa. 4-10}$$

Reference to Equation 2-3 above finds ζ on the left side and in the denominator on the right side of Equation 4-10. Reference to Equation 2-7 …

$$\zeta = (\pi\cdot Z\cdot P_1\cdot U_1\cdot P_2\cdot U_2\cdot\ldots\cdot P_N\cdot U_N)^{1/(N+1)} \qquad \text{Eqa. 2-7}$$

shows that:

$$\zeta^{N+1} = \pi Z\cdot\overline{\overline{PU}} \qquad \text{Eqa. 4-11}$$

Thus Equation 4-10 reduces to an identity confirming our earlier finding that Equation 2-7's determination of ζ does describe the combination of expenditures σ and output ξ that maximizes profits ϕ.

$$\zeta = \left[\frac{\zeta^{N+1}}{\zeta}\right]^{\frac{1}{N}} \qquad \text{Eqa. 4-12}$$

5: Household Utility

The generation of profits must be represented in any faithful model of a capitalistic system. Because SFEcon models presume to comprehend all material and financial flows, they must somehow contrive to have industrial sectors' profits received by some non-industrial sectors. Absent such a construction, there would be no possibility of completing the monetary circuit. Every SFEcon model must therefore contain at least one household sector to receive profits in the form of passive income.

Design of household sectors so as to fit with the general computational scheme describing generic industrial sectors is largely a matter of re-sculpting ideas that have not changed much since Jevons. Our basic premise is that households arrange their affairs for the maximization of leisure; or, more precisely, that time exhausted in the *acquisition* of things is limited by a need to reserve the time needed for the *enjoyment* of things. People generally labor in order to rest; and to earn that which provides comfort, amusement, and security in their leisure time.

Stated formally, this means that one stops working when the enjoyment of a prospective hour of leisure is equal in value to what is earned by the last hour worked. Figure 5-1 sketches such a condition for the case of one person consuming one good. This figure arrays all the parameters developed for industrial productive tradeoffs in Figure 1-1: a set [Z,U] shapes the locus of achievable utility by locating a household utility function's asymptotes; and a price environment [π,P] selects the optimal operating point [ξ,Q]. The 'real wage' is represented by direct intake Q of the sole consumer good. In this example, Q=480 physical units/year is just sufficient to make our consumer content with $-\xi=6766$ hours/year of leisure.

Figure 5-1 introduces a parameter $\tau=8766$ hours/year to express an inescapable limit on each consumer: there are 8766 hours in a year; and all of these hours must be accounted as either labor or leisure. Labor is therefore the residual of τ with ξ: a typical person works about 2000 hours/year, which is τ (= 8766) + ξ (= −6766). Improving economic conditions, allowing the real wage Q to rise, will eventuate in greater leisure and less labor going to market.

SFEcon's sign convention is exercised in Figure 5-1. Labor is positive because it goes into the economy for the sake of producing other things that come back out of the economy. Leisure is negative because it is one of these products: households work to support the consumption needed for contentment within the leisure segment $-\xi$ of their continuing experience of time τ. Figure 5-1 inverts the hyperbolic utility surface's industrial representation in Figure 1-1 by making Z a negative quantity.

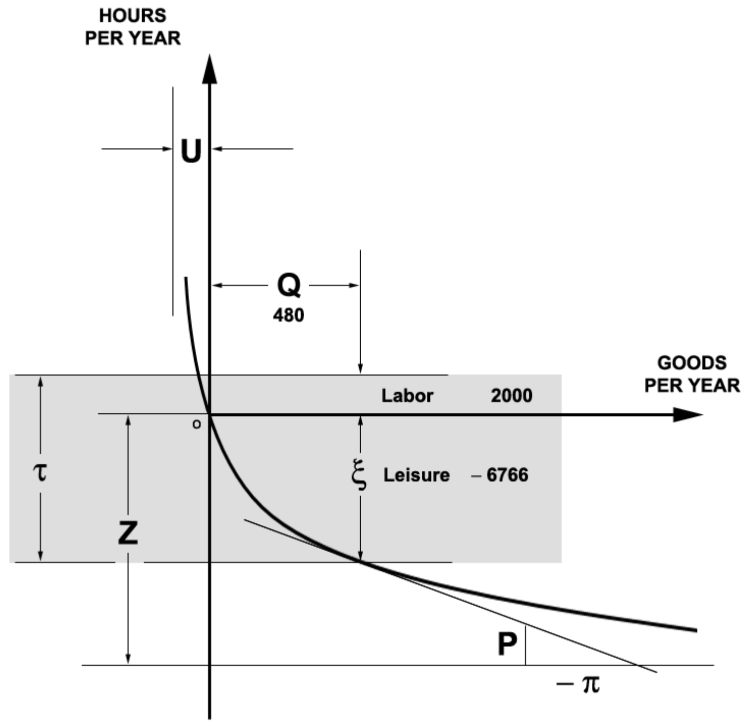

Fig. 5-1

Leisure's money price π is seen operating in the negative relative to the consumable's price P because leisure is a negative quantity: the more one is at rest, the less one earns. According to the premises stated for this analysis (i.e.: the first hour of leisure is equal in value to the last hour of labor) negative π is known because positive π must be the money wage. As $|\pi|$ rises in comparison to commodity prices P, a household can afford to work a bit less and yet consume a bit more; and this marginal increase in consumption will presumably furnish the corresponding increase in leisure.

Households' total cost curve is sketched in Figure 5-2. Note that a negative Z parameter has swung the cost curve over into the negative domain of the horizontal labor/leisure axis. Marginal costs are also negative; and the optimal operating point is selected by equating a negation of the wage to the cost curve's slope. In this example, consumption σ = 47,173 exceeds wages $\pi \cdot (\tau + \xi)$ = 40,000, with the difference being made up out of passive interest income ϕ = -7143. Finally, we note that this formulation allows for a positive ϕ to depict the steady state of a household wherein wages exceed consumption in order to service a constant level of debt.

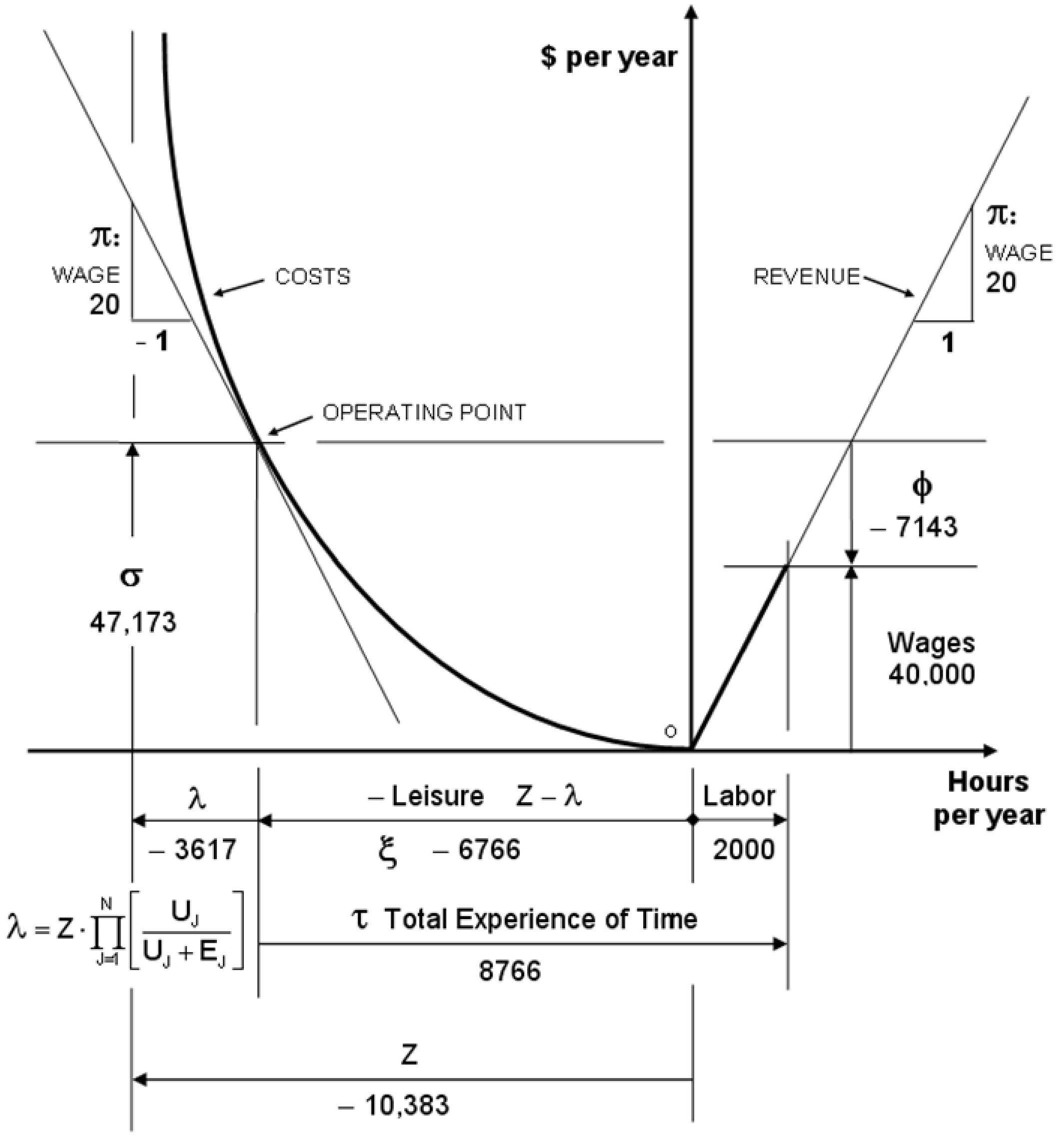

Fig. 5-2

6: Computational Approaches to Optimality

To this point we have surveyed the computational options that are available when prices and the shape of production tradeoffs are known. Let us now consolidate this mathematical development in terms of three strategies for computing economic optima that might prove useful in economic modeling.

As set out in Figure 1-2, the hyperbolic production function relates a physical rate of output Y to a vector of factor employments E_1, E_2 , … , E_N, that are themselves expressed in their respective physical units/year. This functional relationship is controlled by a set of utility parameters Z, U_1, U_2 , … , U_N specifying the shape of an economic actor's production tradeoffs. The essence of production theory is to identify a unique economic optimum ξ, Q_1, Q_2, … , Q_N from among this continuum of technical optima.

Identification of the economic optimum is critically dependent on knowledge of the price environment π, P_1, P_2, … , P_N. For the hyperbolic system, the interaction between prices and the shape of productive options is completely expressed in two parameters:

$$\overline{PU} = \sum_{J=1}^{N} \left(P_J \cdot U_J \right) \qquad \text{Eqa. 3-8}$$

$$\overline{\overline{PU}} = \prod_{J=1}^{N} \left(P_J \cdot U_J \right) \qquad \text{Eqa. 4-3}$$

Knowledge of the price vector identifies an industrial sector's optimal revenues as $\pi \cdot \xi$*; and his budgetary expenditure for asset replenishment as:

$$\sigma = \sum_{J=1}^{N} \left(P_J \cdot Q_J \right) \qquad \text{Eqa. 3-1}$$

Finally, these definitions specify financial services ϕ,

$$\phi = \pi \cdot \xi - \sigma \qquad \text{Eqa. 6-1}$$

the maximization of which defines an economic optimum.

Strategies for computing an economic optimum arise from any of three means by which the optimum might be specified. Once the interactions between the prices P_J of a sector's factors of production and its technical tradeoffs U_J have been established per Equations 3-8 and 4-3, final specification of the optimum can proceed from any one of the set [π, σ, ξ]. Exterior specification of any one of these three parameters should permit a computation of the other two, as well as an estimation of the financial discriminant ζ.

* or $\pi \cdot [\tau + \xi]$ for households. Note that this substitution recurs at obvious places in subsequent algebraic developments.

Figure 6-1 visualizes these three approaches to the cost curve.

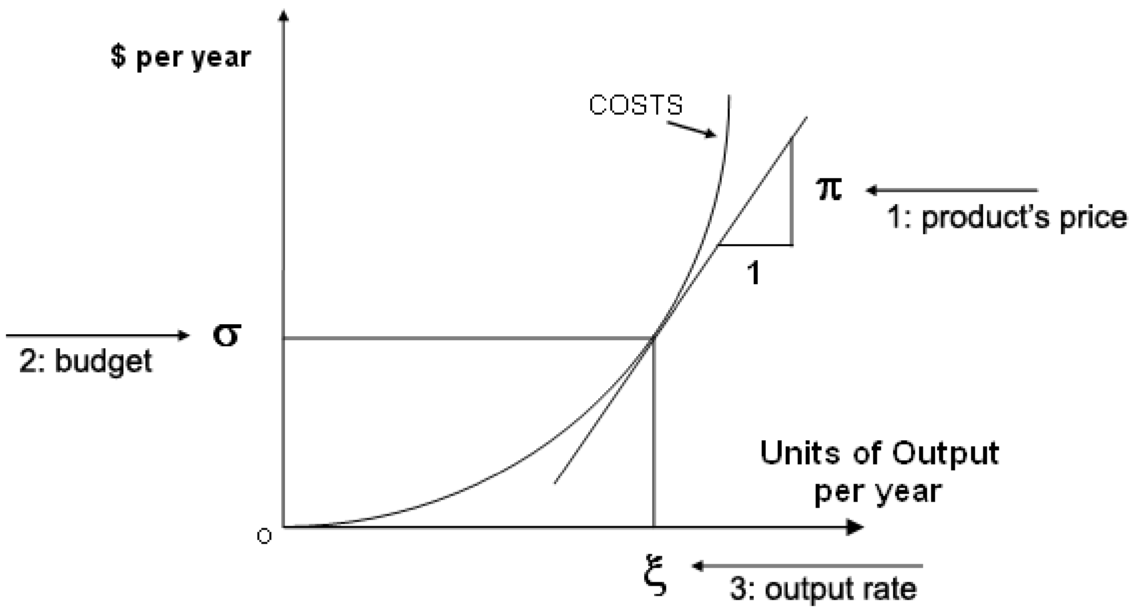

Fig. 6-1

1: When the price π of the product is exogenously specified, as by a perfect market, the parameter ζ is derived from a rearrangement of Equation 4-11:

$$\zeta = \left[\pi \cdot Z \cdot \overline{\overline{PU}}\right]^{\frac{1}{N+1}} \qquad \text{Eqa. 6-2}$$

Optimal output then becomes known from Equation 2-3:

$$\xi = \frac{-\zeta}{\pi} + Z \qquad \text{Eqa. 6-3}$$

and the optimal budget is most easily derived from a rearrangement of Equation 3-9:

$$\sigma = N \cdot \zeta - \overline{PU} \qquad \text{Eqa. 6-4}$$

2: When the computational cycle begins with an exterior specification of the budget σ, the financial discriminant ζ emerges directly from Equation 3-9:

$$\zeta = \frac{\sigma + \overline{PU}}{N} \qquad \text{Eqa. 3-9}$$

The optimal price π of the output derives from a rearrangement of Equation 6-2 above:

$$\pi = \frac{\zeta^{N+1}}{Z \cdot \overline{\overline{PU}}} \qquad \text{Eqa. 6-5}$$

and the optimal ξ output again falls out of Equation 6-3.

3: The final computational approach to optimality begins with a required output rate ξ. Here the optimal budget σ follows directly from the total cost function in Equation 4-6; ζ again falls out of Equation 3-9; and π from Equation 6-5.

Whatever the approach to a cost curve, the optimal financial services ϕ are yielded from Equation 6-1, and the corresponding vector of optimal asset employments Q_J emerge from Equation 3-10:

$$\phi = \pi \cdot \xi - \sigma \qquad \text{Eqa. 6-1}$$

$$Q_J = \frac{\zeta}{P_J} - U_J \qquad \text{Eqa. 3-10}$$

<u>7: Alternate Approaches to ζ</u>

Having completely explored marginalist criteria from the standpoint of an optimal operating decision, we now turn to winnowing-out the marginal values implied by the marginal products at a given point on the surface of technical indifference. When an economic actor's production tradeoffs Z, U_1, U_2, … , U_N are known, any operating state

E_1, E_2, … , E_N will enter the general hyperbolic production function of Equation 1-8 to disclose an output rate Y. All the marginal products would then fall out of Equation 2-1's specification of the hyperbola's gradients; and relative prices would be entirely determined by a re-interpretation of Equation 2-2:

$$\frac{P_J}{\pi} = \frac{Z - Y}{U_J + E_J} \qquad \text{Eqa. 7-1}$$

At this point, most analytic practice concludes by merely observing that, with relative prices known, knowledge of any one price will now determine the absolute magnitude of all prices.

Hyperbolic production functions offer the possibility of further analyses facilitating, and facilitated by, the dynamic modeling context that the hyperbola has been crafted to support. When placed in such a context, hyperbolic production parameters can be used to compute prices and values at their absolute levels, as well as to specify interest rates and currency values throughout multinational I/O models and across time. These models (posted at sfecon.com) are controlled by financial state variables capable of disclosing the appropriate measures of financial services ϕ and budgets σ in a manner that is, as it were, 'conceptually prior' to the computation of prices.

Presuming a sector's ϕ and σ are known, our task here will be to infer absolute marginal values implied by these rates of financial flow. The interactions of ϕ and σ with the marginal products implicit in a physical state defined by the E_J's will determine two estimates of the financial discriminant, θ and β, both of which will equal ζ at a state of optimal equilibrium. (In the disequilibrium states characteristic of dynamic models, θ, β, and ζ, would be seen proceeding by their separate paths toward mutual convergence.)

Taking a sector's budget σ for current asset replenishment as given, θ's approach to ζ derives from the following restatement of a typical equality in Equation 2-3:

$$P_J = \frac{\theta}{U_J + E_J} \qquad \text{Eqa. 7-2}$$

Multiplying both sides of this equation by E_J creates elements of current expenditure on the left side:

$$P_J E_J = \frac{\theta E_J}{U_J + E_J} \qquad \text{Eqa. 7-3}$$

Adding Equations 7-3 for all N Inputs J brings forth the budget σ:

$$\sum_{J=1}^{N} P_J E_J = \sigma = \theta \sum_{J=1}^{N} \frac{E_J}{U_J + E_J} \qquad \text{Eqa. 7-4}$$

Equation 7-4 can then be solved for the desired quantity θ:

$$\theta = \sigma \Big/ \sum_{J=1}^{N} \frac{E_J}{U_J + E_J} \qquad \text{Eqa. 7-5}$$

Taking sector's financial services ϕ as given, β's approach to ζ originates in a sector's earnings ϕ, as defined in Equation 6-1. Substituting current output Y for the optimal ξ in this equation yields:

$$\phi = \pi Y - \sigma \qquad \text{Eqa. 7-6}$$

Equation 2-3's first equality supplies the product's price π in the above:

$$\pi = \frac{\beta}{Z - Y} \qquad \text{Eqa. 7-7}$$

Multiplying both sides of Equation 7-7 by Y supplies the middle term of Equation 7-6:

$$\pi Y = \frac{\beta Y}{Z - Y} \qquad \text{Eqa. 7-8}$$

and substituting β for the θ in Equation 7-5 supplies Equation 7-6's budget term σ:

$$\phi = \frac{\beta Y}{Z - Y} - \beta \sum_{J=1}^{N} \frac{E_J}{U_J + E_J} \qquad \text{Eqa. 7-9}$$

A bit of re-arranging then produces our desired expression for β:

$$\beta = \phi \Big/ \left(\frac{Y}{Z - Y} - \sum_{J=1}^{N} \frac{E_J}{U_J + E_J} \right) \qquad \text{Eqa. 7-10}$$

It must be noted that a slightly different expression for β is required for a household sector. This is because a household's product is leisure time, which is the residual of labor with its total experience of time τ. Remuneration is therefore given by the wage π times $\tau+Y$, where leisure $-Y$ is the negation of a negative quantity Y. Household's equivalent to Equation 7-6 is therefore:

$$\phi = \pi(\tau + Y) - \sigma \qquad \text{Eqa. 7-11}$$

Our sense of Equation 7-7 must also be adjusted to note that the π of this equation computes as a negative to reflect that the marginal cost of leisure is negative: the more leisure a household produces, the less it earns in wages. These considerations enter into an adjusted version of Equation 7-8:

$$\pi(\tau + Y) = -\frac{\beta(\tau + Y)}{Z - Y} \qquad \text{Eqa. 7-12}$$

and must carry into an adjusted version of Equation 7-9:

$$\phi = -\beta\frac{\tau + Y}{Z - Y} - \beta\sum_{J=1}^{N}\frac{E_J}{U_J + E_J} \qquad \text{Eqa. 7-13}$$

Rearranging Equation 7-13 then discloses the household β:

$$\beta = -\phi\Bigg/\left(\frac{\tau + Y}{Z - Y} + \sum_{J=1}^{N}\frac{E_J}{U_J + E_J}\right) \qquad \text{Eqa. 7-14}$$

8: Calibrating the Multi-dimensional Hyperbola

Inference of a sector's hyperbolic description of productive indifference proceeds from an observed operating decision, ξ, Q_1, Q_2, … , Q_N, where the set [ξ,Q] is presumed the optimal instance of the [Y,E] set in Equation 1-8's statement of a production function. Starting from Equation 1-7, we replace the [Y,E] set with [ξ,Q] in anticipation of the algebraic development for the utility set [Z,U].

$$\frac{(Z-\xi)}{Z}=\frac{U_1}{(U_1+Q_1)}\cdot\frac{U_2}{(U_2+Q_2)}\cdot\ \cdots\ \cdot\frac{U_N}{(U_N+Q_N)} \qquad \text{Eqa. 8-1}$$

The [ξ,Q] set is also presumed optimal in regard to an observed price spectrum π, P_1, P_2, … , P_N. Prices [π,P] enter the analysis through a reorganization of Equations 2-3:

$$\begin{aligned} Z - \xi &= \zeta/\pi \\ Z &= \zeta/\pi + \xi \\ U_J + Q_J &= \zeta/P_J \\ U_J &= \zeta/P_J - Q_J \end{aligned} \qquad \text{Eqa. 8-2}$$

Substituting the right-hand sides of Equations 8-2 for their identities in Equation 8-1 eliminates all references to the production coefficients, leaving the financial discriminant ζ as the equation's only unknown:

$$\frac{\zeta/\pi}{\zeta/\pi+\xi}=\frac{\zeta/P_1-Q_1}{\zeta/P_1}\cdot\frac{\zeta/P_2-Q_2}{\zeta/P_2}\cdot\ldots\cdot\frac{\zeta/P_N-Q_N}{\zeta/P_N} \qquad \text{Eqa. 8-3}$$

Cross-multiplying over the equality simplifies Equation 8-3 to ...

$$\frac{\zeta^{N+1}}{\pi\cdot P_1\cdot P_2\cdot\ldots\cdot P_N}=(\zeta/\pi+\xi)\cdot(\zeta/P_1-Q_1)\cdot(\zeta/P_2-Q_2)\cdot\ldots\cdot(\zeta/P_N-Q_N)$$

Eqa. 8-4

Equation 8-4 can be further reduced by multiplying through with the left-hand-side's inverse:

$$1 = (1 + \pi \cdot \xi / \zeta) \cdot (1 - P_1 \cdot Q_1 / \zeta) \cdot (1 - P_2 \cdot Q_2 / \zeta) \cdot \ldots \cdot (1 - P_N \cdot Q_N / \zeta) \qquad \text{Eqa. 8-5}$$

Extracting ζ from this equation begins by taking a natural logarithm of each side:

$$0 = \ln(1 + \pi \cdot \xi / \zeta) + \ln(1 - P_1 \cdot Q_1 / \zeta) + \ln(1 - P_2 \cdot Q_2 / \zeta) + \ldots + \ln(1 - P_N \cdot Q_N / \zeta) \qquad \text{Eqa. 8-6}$$

Solving Equation 8-6 requires reference to the series expansion of the natural logarithm. When | a | < 1,

$$\ln(1 + a) = a - a^2/2 + a^3/3 - a^4/4 + \ldots \qquad \text{Eqa. 8-7}$$

Stating Equation 8-6 in terms of this expansion leads to ...

$$\begin{aligned} 0 = {} & \frac{(\pi \cdot \xi)}{\zeta} - \frac{(\pi \cdot \xi)^2}{2 \cdot \zeta^2} + \frac{(\pi \cdot \xi)^3}{3 \cdot \zeta^3} - \frac{(\pi \cdot \xi)^4}{4 \cdot \zeta^4} + \cdots \\ & + \frac{(-P_1 \cdot Q_1)}{\zeta} - \frac{(-P_1 \cdot Q_1)^2}{2 \cdot \zeta^2} + \frac{(-P_1 \cdot Q_1)^3}{3 \cdot \zeta^3} - \frac{(-P_1 \cdot Q_1)^4}{4 \cdot \zeta^4} + \cdots \\ & + \frac{(-P_2 \cdot Q_2)}{\zeta} - \frac{(-P_2 \cdot Q_2)^2}{2 \cdot \zeta^2} + \frac{(-P_2 \cdot Q_2)^3}{3 \cdot \zeta^3} - \frac{(-P_2 \cdot Q_2)^4}{4 \cdot \zeta^4} + \cdots \\ & \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \cdots \\ & + \frac{(-P_N \cdot Q_N)}{\zeta} - \frac{(-P_N \cdot Q_N)^2}{2 \cdot \zeta^2} + \frac{(-P_N \cdot Q_N)^3}{3 \cdot \zeta^3} - \frac{(-P_N \cdot Q_N)^4}{4 \cdot \zeta^4} + \cdots \end{aligned}$$

Eqa. 8-8

Taking the first four terms in each expansion of Equation 8-8 to approximate the equality and multiplying through by ζ^4 brings us to Equation 8-9, which is a soluble, cubic equation in ζ for which all the coefficients are observed among an economic sector's operating decisions.

$$
\begin{aligned}
0 = \; & \frac{\zeta^3}{1} \cdot \left[+ (\pi \cdot \xi) \; + \; (-P_1 \cdot Q_1) \; + \; (-P_2 \cdot Q_2) \; + \; \cdots \; + \; (-P_N \cdot Q_N) \right] \\
+ \; & \frac{\zeta^2}{2} \cdot \left[- (\pi \cdot \xi)^2 - (-P_1 \cdot Q_1)^2 - (-P_2 \cdot Q_2)^2 - \cdots - (-P_N \cdot Q_N)^2 \right] \\
+ \; & \frac{\zeta}{3} \cdot \left[+ (\pi \cdot \xi)^3 + (-P_1 \cdot Q_1)^3 + (-P_2 \cdot Q_2)^3 + \cdots + (-P_N \cdot Q_N)^3 \right] \\
+ \; & \frac{1}{4} \cdot \left[- (\pi \cdot \xi)^4 - (-P_1 \cdot Q_1)^4 - (-P_2 \cdot Q_2)^4 - \cdots - (-P_N \cdot Q_N)^4 \right]
\end{aligned}
$$

Eqa. 8-9

A quadratic approximation is also available on the basis of Equation 8-8's first three terms. These two approximations to the financial discriminant ζ, and knowledge of which is the better of the two estimates, allows formulation of any number of iterative processes by which ζ might be reported to any desired accuracy. Once ζ has been extracted from this system, Z and all the U 's fall out of Equations 8-2.

9: Directions

While the fundamental presumption of this monograph has been the optimality of an economic sector's steady-state, we have made scattered references to the dynamic, multi-sectoral, macroeconomic models that are supported by hyperbolic descriptions of productive indifference. Some flavor of these models can be seen in Figure 9-1, where one of the earliest BEA benchmark I/O tables has been consolidated and reorganized according to the needs of such a model.

Rows correspond to economic sectors, and corresponding columns to the commodities they produce. Row 0 contains the negative sense of output ξ; and Column 0 receives the negative of sectors' budgets σ. Because these data are compiled in monetary units, we must presume that all prices are unity, and that each commodity is measured in whatever physical unit this happens to imply.

The model's operation can be envisioned as an emulation of time in terms of a continuous regeneration of this matrix, with the initial unitary prices (really price indexes) varying as the model seeks its general optimum. Such simulations are, of course, only meaningful if they comprise distinct tables for each national economy in the global web of trade. Since all such tables must have identical definitions of sector and commodity, they must conform to something like Figure 9-1's high degree of consolidation.

The boundary conditions for these models would be the hyperbolic production coefficients of the sectors, which might be organized along the lines of Figure 9-2. Here each sector's production coefficients U_J are set out in rows. The parameters Z are in Row 0 at the column index corresponding to a sector's row. Column 0 receives each sector's $-\overline{PU}$. Parameters for Sectors 1 through 18, the producing sectors in this model, have been constructed according to the formulae set out in Article 8. Sectors19 and 20 contain, respectively the Household and Government Sectors, and require somewhat different calculations.

In its application to very large scale, long range dynamic systems, the hyperbola offers the initial advantage of its computational compactness, i.e.: its closed-form disclosures of critical economic references. Hyperbolae are also more transparent to the purposes of economic theory than other functional forms – a matter most apparent in the hyperbola's ability to express a varying relation between average products and marginal products (unlike the Cobb-Douglas production function, which permanently fixes this relation).

The chief advantage of hyperbolae is most likely to issue from this function's unique relationship with dynamic phenomena generally. In its expression of technical optima, a production function relates inputs to an output rate; and production functions also interact with price levels to describe optimal input rates, which add up to the demand rates for each economic good. When set in a dynamic context, these computations of supply and demand will naturally have their differences continuously integrated to establish market levels for every good. Control of these levels will actuate the price adjustments that, in turn, control the entire model. Since the integral under a hyperbolic surface is a natural logarithm, this simple strategy will install the number e at the center of any dynamic analysis. Economic models based on hyperbolic production surfaces would therefore automatically engage the notions of Fibonacci and Taylor Series that somehow always seem to underlie all numerically precise representations of dynamic phenomena.

Million $
USA 1977

	(27,056)	11,398	(4,202)	(7,929)	(4,274)	506	(35,283)	(7,566)	(735)	20
PE	0 CONTROL	1 AGRCLTR	2 FOOD_&T	3 TEXTILE	4 FRST_PR	5 MINING_	6 PETRO_X	7 CHEMICL	8 CERAMIC	9 CNSTRTN
0 CONTROL	5,012,472	(120,967)	(205,632)	(93,569)	(95,818)	(28,419)	(43,211)	(256,146)	(34,798)	(265,509)
1 AGRCLTR	(92,839)	31,565	11,401	268	496	162		11,992	77	1,091
2 FOOD_&T	(182,252)	54,740	40,744	77	7,606	116		5,097	3,111	983
3 TEXTILE	(89,076)	2,112	557	36,496	1,179	42		10,973	132	425
4 FRST_PR	(82,130)	516	304	869	32,937	339		6,661	327	1,094
5 MINING_	(22,997)	6	2	44	144	3,270		1,518	106	208
6 PETRO_X	(20,982)	2	5	13	5		2,436	728	18	2,721
7 CHEMICL	(228,940)	346	1,241	1,785	4,011	2,583	62,471	61,907	1,239	2,147
8 CERAMIC	(30,006)	4	21	140	1,193	1,989		2,184	4,339	558
9 CNSTRTN	(248,839)	662	10	1,498	19,606	2,036		14,396	16,789	341
10 PRIMARY	(182,422)	3	22	104	1,316	9,121	27	7,587	1,180	2,714
11 HVY_MFG	(70,217)	1	12	114	411	17		1,869	551	414
12 LHT_MFG	(217,507)	23	113	5,179	5,066	93		12,524	2,317	893
13 ELCTRNX	(105,528)	2	15	92	1,316	38		6,152	1,216	562
14 TRNSPRT	(108,133)	8	73	244	133	3	80	10,085	75	3,259
15 CMMRC&F	(542,180)	2,126	338	364	3,886	5		7,806	197	17,987
16 CMMNCTN	(85,033)	1	19	140	8,410	3		1,928	29	2,168
17 SERVICE	(400,069)	2,250	27,732	2,105	3,582	17		14,091	1,293	4,679
18 UTILTYS	(77,524)	7	5	5	108	6,043	13,263	8,726	9	3,716
19 HSHLD__	(1,609,022)	10,655	123,103	50,743	7,002	609	117	65,543	2,001	158,806
20 GVRNMNT	(643,832)	4,540	4,117	1,218	1,685	1,427	100	11,945	527	60,723

Figure 9-1: Consolidated Input/Output Table for the U.S. Economy in 1977

(6,820)	6,688	(2,950)	243	6,568	24,021	376	3,592	(1,929)	(8,973)	193

10 PRIMARY	11 HVY_MFG	12 LHT_MFG	13 ELCTRNX	14 TRNSPRT	15 CMMRC&F	16 CMMNCTN	17 SERVICE	18 UTILTYS	19 HSHLD__	20 GVRNMNT	
(198,805)	(81,291)	(233,104)	(115,776)	(128,315)	(792,280)	(85,424)	(530,871)	(122,456)	(1,181,378)	(398,703)	0
303	933	153	421	1,901	13,712	349	2,664	1,610	11,063	2,678	1
7,005	287	66	52	4,582	11,874	1,430	6,655	2,136	26,490	9,201	2
153	501	738	51	1,027	4,849	648	3,046	1,245	24,033	869	3
1,844	627	297	84	2,924	5,774	250	2,393	2,301	21,136	1,453	4
1,079	1,708	136	163	311	1,742	41	1,179	1,226	9,221	893	5
923	575	43	278	210	5,209	76	1,215	767	3,570	2,188	6
5,120	1,460	374	169	8,256	10,274	876	11,646	7,787	36,799	8,449	7
749	402	156	58	2,152	1,664	161	1,289	1,775	10,505	667	8
34,003	2,388	1,998	10,673	5,934	26,350	1,065	16,911	850	90,372	2,957	9
66,304	4,992	415	1,380	5,658	14,630	801	5,132	5,954	52,442	2,640	10
15,699	10,067	662	2,121	1,004	4,821	457	2,737	851	27,453	956	11
37,576	4,393	44,370	10,073	3,253	12,474	994	9,282	1,964	63,812	3,108	12
14,878	1,659	1,207	20,284	1,674	8,221	791	5,375	1,219	39,593	1,234	13
1,081	472	2,327	396	19,867	5,671	1,266	7,831	1,074	49,595	4,593	14
541	439	1,684	915	8,006	79,999	12,195	62,425	11,044	221,353	110,870	15
399	315	962	1,801	1,658	4,946	7,663	8,488	927	37,770	7,406	16
3,792	1,670	10,784	4,066	6,951	42,197	11,659	45,952	8,471	196,845	11,933	17
792	84	163	211	2,069	2,344	393	1,498	20,882	11,999	5,207	18
10,166	39,823	141,876	52,368	35,336	495,636	36,796	308,924	42,416	20,834	6,268	19
3,218	1,808	27,643	9,969	8,974	15,872	7,137	22,637	9,886	235,466	214,940	20

Figure 9-1 (continued)**:** Consolidated Input/Output Table for the U.S. Economy in 1977

Million $
USA 1977

PU	0 CONTROL	1 AGRCLTR	2 FOOD_&T	3 TEXTILE	4 FRST_PR	5 MINING_	6 PETRO_X	7 CHEMICL	8 CERAMIC	9 CNSTRTN
	1	1	1	1	1	1	1	1	1	1
0 CONTROL		344,755	1,122,115	1,253,411	443,231	98,224	62,775	1,500,193	158,247	2,578,709
1 AGRCLTR	(4,159,140)	192,223	212,387	223,520	223,292	223,626	0	211,796	223,711	222,697
2 FOOD_&T	(17,230,923)	861,743	875,739	916,406	908,877	916,367	0	911,386	913,372	915,500
3 TEXTILE	(21,947,916)	1,157,730	1,159,285	1,123,346	1,158,663	1,159,800	0	1,148,869	1,159,710	1,159,417
4 FRST_PR	(6,518,721)	346,897	347,109	346,544	314,476	347,074	0	340,752	347,086	346,319
5 MINING_	(1,303,294)	69,799	69,803	69,761	69,661	66,535	0	68,287	69,699	69,597
6 PETRO_X	(350,740)	19,562	19,559	19,551	19,559	0	17,128	18,836	19,546	16,843
7 CHEMICL	(24,652,005)	1,243,701	1,242,806	1,242,262	1,240,036	1,241,464	1,181,576	1,182,140	1,242,808	1,241,900
8 CERAMIC	(2,315,524)	123,445	123,428	123,309	122,256	121,460	0	121,265	119,110	122,891
9 CNSTRTN	(43,701,975)	2,312,538	2,313,190	2,311,702	2,293,594	2,311,164	0	2,298,804	2,296,411	2,312,859
10 PRIMARY	(26,594,021)	1,338,819	1,338,800	1,338,718	1,337,506	1,329,701	1,338,795	1,331,235	1,337,642	1,336,108
11 HVY_MFG	(5,798,790)	308,894	308,883	308,781	308,484	308,878	0	307,026	308,344	308,481
12 LHT_MFG	(35,362,734)	1,872,621	1,872,531	1,867,465	1,867,578	1,872,551	0	1,860,120	1,870,327	1,871,751
13 ELCTRNX	(13,358,735)	708,643	708,630	708,553	707,329	708,607	0	702,493	707,429	708,083
14 TRNSPRT	(8,345,933)	422,695	422,630	422,459	422,570	422,700	422,623	412,618	422,628	419,444
15 CMMRC&F	(18,454,598)	997,704	999,492	999,466	995,944	999,825	0	992,024	999,633	981,843
16 CMMNCTN	(218,303,814)	11,494,148	11,494,130	11,494,009	11,485,739	11,494,146	0	11,492,221	11,494,120	11,491,981
17 SERVICE	(18,493,186)	992,131	966,649	992,276	990,799	994,364	0	980,290	993,088	989,702
18 UTILTYS	(2,112,014)	109,469	109,471	109,471	109,368	103,433	96,213	100,750	109,467	105,760
19 HSHLD__	(22,018,557)	1,170,723	1,058,275	1,130,635	1,174,376	1,180,769	1,181,261	1,115,835	1,179,377	1,022,572
20 GVRNMNT	(7,330,248)	394,163	394,586	397,485	397,018	397,276	398,603	386,758	398,176	337,980

Figure 9-2**:** Hyperbolic Production Coefficients for the Consolidated 1977 Table

1	1	1	1	1	1	1	1	1	1	1

10 PRIMARY	11 HVY_MFG	12 LHT_MFG	13 ELCTRNX	14 TRNSPRT	15 CMMRC&F	16 CMMNCTN	17 SERVICE	18 UTILTYS	19 HSHLD__	20 GVRNMNT	
1,537,627	390,186	2,105,748	824,421	551,018	1,792,110	11,579,573	1,525,252	231,932	1,181,378	398,703	0
223,485	222,855	223,635	223,367	221,887	210,076	223,439	221,124	222,178	212,725	221,110	1
909,478	916,196	916,417	916,431	911,901	904,609	915,053	909,828	914,347	889,993	907,282	2
1,159,689	1,159,341	1,159,104	1,159,791	1,158,815	1,154,993	1,159,194	1,156,796	1,158,597	1,135,809	1,158,973	3
345,569	346,786	347,116	347,329	344,489	341,639	347,163	345,020	345,112	326,277	345,960	4
68,726	68,097	69,669	69,642	69,494	68,063	69,764	68,626	68,579	60,584	68,912	5
18,641	18,989	19,521	19,286	19,354	14,355	19,488	18,349	18,797	15,994	17,376	6
1,238,927	1,242,587	1,243,673	1,243,878	1,235,791	1,233,773	1,243,171	1,232,401	1,236,260	1,207,248	1,235,598	7
122,700	123,047	123,293	123,391	121,297	121,785	123,288	122,160	121,674	112,944	122,782	8
2,279,197	2,310,812	2,311,202	2,302,527	2,307,266	2,286,850	2,312,135	2,296,289	2,312,350	2,222,828	2,310,243	9
1,272,518	1,333,830	1,338,407	1,337,442	1,333,164	1,324,192	1,338,021	1,333,690	1,332,868	1,286,380	1,336,182	10
293,196	298,828	308,233	306,774	307,891	304,074	308,438	306,158	308,044	281,442	307,939	11
1,835,068	1,868,251	1,828,274	1,862,571	1,869,391	1,860,170	1,871,650	1,863,362	1,870,680	1,808,832	1,869,536	12
693,767	706,986	707,438	688,361	706,971	700,424	707,854	703,270	707,426	669,052	707,411	13
421,622	422,231	420,376	422,307	402,836	417,032	421,437	414,872	421,629	373,108	418,110	14
999,289	999,391	998,146	998,915	991,824	919,831	987,635	937,405	988,786	778,477	888,960	15
11,493,750	11,493,834	11,493,187	11,492,348	11,492,491	11,489,203	11,486,486	11,485,661	11,493,222	11,456,379	11,486,743	16
990,589	992,711	983,597	990,315	987,430	952,184	982,722	948,429	985,910	797,536	982,448	17
108,684	109,392	109,313	109,265	107,407	107,132	109,083	107,978	88,594	97,477	104,269	18
1,171,212	1,141,555	1,039,502	1,129,010	1,146,042	685,742	1,144,582	872,454	1,138,962	1,160,544	1,175,110	19
395,485	396,895	371,060	388,734	389,729	382,831	391,566	376,066	388,817	163,237	183,763	20

Figure 9-2 (continued): Hyperbolic Production Coefficients for the Consolidated 1977 Table

GLOSSARY:

- The production function relates ...

Y = the rate of production, units/yr, to
E_J = the rates of asset expenditure, units/yr

Z, U_J, N } by way of utility parameters expressing the shape of an economic actor's production alternatives. (N is the number of productive factors)

- Economic optimality makes critical references to prices:

π = the price of the good being produced, \$/unit, and
P_J = the prices of the assets used in production.

- Two parameters express the interaction between asset prices and utility parameters:

$$\overline{PU} = \sum_{J=1}^{N} \left(P_J \cdot U_J \right)$$

$$\overline{\overline{PU}} = \prod_{J=1}^{N} \left(P_J \cdot U_J \right)$$

- An economic actor's pattern of money expenditure divides between ...

σ = \$/yr for productive inputs, and
ϕ = \$/yr for financial services, dividends, etc.

- Economic optimality (maximum ϕ) occurs at:

ξ = units/yr of output, supported by
Q_J = units/yr of factor employments.